The existence of homomorphisms to oriented cycles

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Abstract

We discuss the existence of homomorphisms of arbitrary digraphs to a fixed oriented cycle C. Our main result asserts that if the cycle C is unbalanced then a digraph G is homomorphic to C if and only if (1) every oriented path homomorphic to G is also homomorphic to C, and (2) the length of every cycle of G is a multiple of the length of C. This answers a conjecture from an earlier paper with H. Zhou, and generalizes a result proved there. We also show that this characterization does not hold for balanced cycles. We relate these results to work on the complexity of homomorphism problems.

1 Introduction

All the digraphs discussed in this paper are finite unless otherwise specified. A *homomorphism* $G \to H$ of a digraph G to a digraph H is a mapping of

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the vertex sets $f : V(G) \to V(H)$ which preserves the edges, i.e., such that $xy \in E(G)$ implies $f(x)f(y) \in E(H)$. If such a homomorphism exists, we say G is *homomorphic* to H and write $G \to H$. Otherwise we write $G \not\to H$. Note that these notions can also be applied to undirected graphs, by viewing them as symmetric digraphs. If K_n denotes the undirected complete graph on n vertices, then for an arbitrary undirected graph G a homomorphism $G \to K_n$ is just an n-colouring of G. Because of this fact, it is also common to call a homomorphism $G \to H$ (of general digraphs) an *H-colouring* of G.

Suppose $g_0 \in V(G)$ is a fixed vertex of G, called the root of G, and suppose $h_0 \in V(H)$ is the root of H. A *rooted homomorphism* of G to H is a homomorphism $h : G \to H$ such that $h(g_0) = h_0$. In this case we write $(G, g_0) \to (H, h_0)$ and say h is a homomorphism of (G, g_0) to (H, h_0). We observe that the composition of rooted homomorphisms $(G, g_0) \to (H, h_0)$ and $(H, h_0) \to (J, j_0)$ is a rooted homomorphism $(G, g_0) \to (J, j_0)$.

An *oriented path* P is a sequence of distinct vertices $[p_0, p_1, \ldots, p_n]$, such that, for each $i \in \{0, 1, \ldots, n-1\}$, either $p_ip_{i+1} \in E(P)$ (a *forward* edge of P), or $p_{i+1}p_i \in E(P)$ (a *backward* edge of P), and such that P has no other edges. The direction in which P is traversed is emphasized by saying that p_0 is the *initial vertex*, $i(P)$, of P, and p_n the *terminal vertex*, $t(P)$, of P, respectively.

An *oriented cycle* C is a digraph obtained from an oriented path P by identifying its initial and terminal vertices. Thus an oriented cycle C can be given by a circular sequence of vertices $[c_0, c_1, \ldots, c_{m-1}, c_0]$, such that, for each $i \in \{0, 1, \ldots, m-1\}$, either $c_ic_{i+1} \in E(C)$ (a *forward* edge of C), or $c_{i+1}c_i \in E(C)$ (a *backward* edge of C), and such that C has no other edges. (Subscript addition is taken modulo m.) The direction of C which agrees with the forward edges, i.e., $c_0, c_1, c_2, \ldots$, is the *positive direction* of C, and the opposite direction, i.e., $c_0, c_{m-1}, c_{m-2}, \ldots$, is the *negative direction* of C. We also use the term *path* (resp. *cycle*) to mean an oriented path (resp. an oriented cycle). Since we do not distinguish an initial vertex of an oriented cycle, $[c_0, c_1, \ldots, c_{m-1}, c_0] = [c_1, c_2, \ldots, c_{m-1}, c_0, c_1]$, and we usually choose a most convenient vertex to start listing C.

A *directed cycle* (resp. a *directed path*) is a cycle (resp. path) in which all