A SPLITTING PRINCIPLE

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ABSTRACT. A new technical tool for K-theoretical computation is given. It is a "splitting principle". It is applied to compute K-groups of any principal G-homogeneous space X, where G is an arbitrary semisimple simply connected algebraic group over a field F. The answer is given in terms of K-groups of certain central simple F-algebras depending only of G. In particular we prove the isomorphism $K_0(F[X]\otimes\mathcal{D})\cong K_0(\mathcal{D})$ for any finite dimensional separable F-algebra $\mathcal{D}$.

INTRODUCTION

The present paper has two main goals. The first one is to give a new technical tool for certain K-theoretical computations (Theorem 1.1). It is a "splitting principle". The second goal is to apply this "splitting principle" to compute K-groups of any principal G-homogeneous space X, where G is an arbitrary semimisimple and simply connected algebraic group (Theorem 5.1). (Actually G is subject to a condition (see §5), which we omit in the introduction.) For K_0-groups Theorem 5.1 states the isomorphism

$$K_0(\mathcal{D})\cong K_0(F[X]\otimes_F\mathcal{D}) \tag{*}$$

for any finite-dimensional separable F-algebra $\mathcal{D}$.

Consequences of isomorphisms $(*)$ are the injectivity of the Brauer group homomorphism $\mathrm{Br}(F)\to\mathrm{Br}(F(X))$ and the triviality of the first and the second Chow groups of X (see 5.2).

Actually we compute higher K-groups of mentioned varieties as well. The corresponding result is also stated in 5.1. It expresses K-groups of X in terms of K-groups of certain central simple F-algebras, which are listed in 5.4. Theorem 5.1 generalizes a theorem of Suslin ([Su], 6.3) and a theorem of Levine ([L], 4.3), where K-groups of certain norm varieties are computed.

The model case of Theorem 5.1 is the following result due to Levine ([L], 2.3):

$$\coprod_I K_*(F)[-\operatorname{card}(I)]\cong K_*(G)$$

for every semisimple and simply connected F-split algebraic group G over a field F, where the sum runs over all subsets I of the set $\{1,2,\dots,\mathrm{rk}(G)\}$. Our "splitting principle" reduces the general case of Theorem 5.1 to this model case (see 10.1, 10.2).

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This paper consists of two parts. The first one is §§1 − −4. It is devoted to the "splitting principle". The second one is §§5 − −14. It is devoted to the K-theory of principal homogeneous spaces. The main result of the first part is Theorem 1.1. The main result of the second part is Theorem 5.1.

To be more specific, in §1 we state the "splitting principle" (Theorem 1.1) and prove a more technical result (see 1.4). In §2, we recall the relations $f_*[\mathcal{O}_{Y\times S}] = [\mathcal{O}_S]$ for a certain class of varieties Y, and prove the triviality of any principal B-homogeneous space for any Borel subgroup B of any semisimple algebraic group. In §3, we introduce the notions of K-module and K-module morphism (see 3.2, 3.4), discuss their properties (3.3, 3.5), and consider examples (3.6, 3.7). In §4, we prove the splitting principle (1.1) using results of §2 and prove Theorem 1.4 using above relations $f_*[\mathcal{O}_{Y\times S}] = [\mathcal{O}_S]$.

In §5, the Main Theorem (see 5.1) is stated. The list of algebras is given in 5.4, and the map from 5.1 is described in 5.3. In §6 we recall the Chevalley theorem about the representation ring of semisimple algebraic group (see 6.1), then we define a notion of fundamental family of representations (see 6.2) and consider examples (see 6.3). In §7, Levine's result is stated (see 7.2) and a modification of it is proved (see 7.6). In §8, we describe Tits' construction of central simple F-algebras arising ¿from representations (see 8.5) and discuss its properties (see 8.7–8.10). In §9 we state variants of Theorem 5.1 using Tits' construction and the fundamental family of representations (see 9.3, 9.5, 9.7). In §10 we prove Theorems 9.5 and 9.7 using 1.1 and 1.4. In §§11, 12, 13 technical lemmas used in §10 are proved. In §14 relations between Clifford algebras and spinor representations are discussed (see 14.8). In §15, we deduce Theorem 5.1 from 9.7 using 8.7 and 13.8.

Throughout the present paper we shall use the following notation and conventions:

F is a field,

$\mathcal{D}$ is a separable F-algebra,

$\mathcal{D}^{\mathrm{op}}$ is the opposite F-algebra.

The word "scheme" means a separate noetherian F-scheme of finite Krull's dimension, and if X is a scheme, then

$\mathcal{P}(X)$ is the category of locally free coherent $\mathcal{O}_X$-modules,

$\mathcal{M}(X)$ is the category of coherent $\mathcal{O}_X$-modules,

$\mathcal{M}(X, \mathcal{D}^{\mathrm{op}})$ is the category of coherent right $\mathcal{D} \otimes_F \mathcal{O}_X$-modules,

$\mathcal{P}(X, \mathcal{D}^{\mathrm{op}})$ is the category of locally projective coherent right $\mathcal{D} \otimes_F \mathcal{O}_X$-modules, or, what is the same, the category of locally free coherent $\mathcal{O}_X$-modules equipped with right $\mathcal{D}$-action and morphisms compatible with this $\mathcal{D}$-action.

Let $K_i(\mathcal{C}) := \pi_{i+1}(BQ\mathcal{C})$ for any exact category $\mathcal{C}$, where $Q\mathcal{C}$ is the category obtained by Quillen's Q-construction ([Q],§2), and $BQ\mathcal{C}$ denotes the classifying space of $Q\mathcal{C}$. As usual, let

$$
\begin{aligned}
K_i(X) &= K_i(\mathcal{P}(X)),\\
K'_i(X) &= K_i(\mathcal{M}(X)),\\
K_i(X, \mathcal{D}^{\mathrm{op}}) &= K_i(\mathcal{P}(X, \mathcal{D}^{\mathrm{op}})),\\
K'_i(X, \mathcal{D}^{\mathrm{op}}) &= K_i(\mathcal{M}(X, \mathcal{D}^{\mathrm{op}}).
\end{aligned}
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