Splitting Patterns of Quadratic Forms

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Introduction.

Let q be an anisotropic quadratic form of dimension $n \geq 2$ over a field k of characteristic $\neq 2$. The splitting pattern of q is made up from all distinct Witt indices of q which occur over field extensions of k. Those are called the higher Witt indices of q.

For excellent quadratic forms q, the splitting pattern depends only on the dimension n of q, and is known explicitly for every n. This goes back to the originating work of M. Knebusch [10, 11] and has been studied further in [5]. However, surprisingly little is known about the splitting pattern of quadratic forms q, in general, as soon as there are at least 2 non-zero Witt indices in the splitting pattern of q.

In section 1 we investigate general constraints on splitting patterns. In theorem 1.6 we describe in which way the splitting pattern of a quadratic form is influenced by its Clifford algebra. For this we use the index reduction formulas of A. S. Merkurjev [13] as the crucial tool. It turns out that the splitting of the Clifford algebra determines the "high end" of the splitting pattern.

On its "low end", we find that the first higher Witt index $i_1(q)$ of q always is less than or equal to the first higher Witt index of the anisotropic excellent forms of the same dimension as q. In particular, $i_1(q)$ is less than or equal to the excess of $\dim q$ over the biggest 2-power which is strictly smaller than $\dim q$. If equality holds, then q is a "stable Pfister neighbor", that is, there is a field extension K of k such that q_K is an anisotropic Pfister neighbor.

Similar results on $i_1(q)$ have earlier been obtained by D. W. Hoffmann [4].

Here we give a short proof based on Theorem 1.7, which describes a useful criterion to determine when an anisotropic form q is a summand of an anisotropic Pfister form π. Corollary 1.8 establishes the existence of a field extension K of k such that π_K is anisotropic and has the anisotropic kernel $(q_K)_{\mathrm{an}}$ of q as an orthogonal summand, and it surprisingly turns out that the index $i(q_K)$ depends only on q and π, not on K.

Special cases of our corollaries 1.9, 1.11, 1.12, 1.13 are to be found in [4, cf. the "Main Lemma", Thms. 1, 2 and corollaries].

In section 2 we study the "algebraic" splitting pattern; that is, we look at the possible Witt indices which can occur over algebraic extensions of k. In general, the algebraic splitting pattern is different from the "absolute" splitting pattern. However, we show that for given dimension n there is a number t such that, for every form q of dimension n over any field k, the splitting pattern of q is equal to the algebraic splitting pattern of q_K, where K is a purely transcendental field extension of k of degree t.

In section 3 we approach the wide open case of quadratic forms of height 2; that is, with exactly 2 non zero Witt indices in the splitting pattern of q; see, in particular, theorems 3.4 and 3.11 for so called good forms of degree 3 and forms of degree 2 that are not good, respectively. The general theorem 3.9 makes explicit the splitting pattern of good forms of arbitrary degree, assuming that the conjectures by B. Jacob

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and M. Rost [6] on higher cohomological invariants are true. Under this assumption those forms have always twice the dimension of their leading form, and over a suitable field extension they are (still anisotropic) a product of a four dimensional form and a Pfister form. This also (hypothetically) answers a question of M. Knebusch [11, 10.6, p. 30].

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0. Setup.

Let q be a regular quadratic form of index $i(q)$ and let

$$i_0 := i(q) < i_1 < \ldots < i_{h-1} < i_h = [n/2]$$

denote all distinct Witt indices which occur for the quadratic forms $q_L := q \otimes L$ with L running through all field extensions of k.

The natural number h is called the *height* of q. The $(h+1)$-tuple $(i_0, i_1, \ldots, i_h)$ of strictly increasing integers is called the *splitting pattern of* q (cf. [5, 1.1]). If necessary, we will write $i_\nu(q)$ and $h(q)$ instead of i_ν and h. We now illustrate the extreme cases $h = 1$ and $h = m := [n/2]$.

0.1 Example (Height 1; cf. [10, Thm. 5.8]). *An anisotropic quadratic form q of dimension ≥ 2 over k has splitting pattern $(0, m)$ for some $m \in \mathbb{N}$ if and only if q is similar to an orthogonal summand of codimension ≤ 1 of a Pfister form over k.*

0.2 Example (Height m; cf. [10, Ex. 5.7]). *Let $q = \langle X_1, X_2, \ldots, X_n \rangle$ with indeterminates X_i over $k = F(X_1, \ldots, X_n)$ for any field F. Then q has splitting pattern*

$$(0, 1, 2, \ldots, m-2, m-1, m).$$

Given $m \in \mathbb{N}$, there are 2^{m-1} different tuples $(0, \ldots, m)$ of strictly increasing integers. What about realizing them as splitting patterns? By example 0.1, there is *no* quadratic form q with splitting pattern $(0, 5)$ or $(0, 6)$, say, since there do not exist Pfister forms of appropriate dimensions.

The basic question of our paper is: What are the splitting patterns of quadratic forms?

The above observation concerning splitting patterns of quadratic forms of height 1 yields immediately that a tuple $(0, \ldots, m-a, m)$ can be realized to be a splitting pattern of some quadratic form of arbitrary height *only if* a or $a-1$ is a 2-power.

We will look for further restrictions on splitting patterns and will comment on the next-to-extreme cases $h = 2$ and $h = m-1$.

1. General constraints on splitting patterns

We will present an analogon to example 0.2. Consider the rational function field $k = F(X_1, \ldots, X_{n-1})$ with indeterminates $X_1, \ldots X_{n-1}$ for any field F, let $c \in F^*$.

1.1 Lemma. *The n-dimensional quadratic form*

$$q = \langle X_1, X_2, \ldots, X_{n-1}, cX_1X_2 \cdots X_{n-1} \rangle$$