Generalized Weyl algebras

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1 Introduction

The purpose of this paper is to study the simple modules over one class of generalized Weyl algebras (briefly, GWA, the definition is in sec. 3), containing the classical Weyl algebras A_n. In sec. 4 the classification (up to irreducible elements of some Euclidian ring) of simple modules over GWA of degree 1 is considered. As particular case the classification of simple $sl(2)$-modules is obtained. The present method of approach adopt R. Block's idea to use some localization functor for the classification of the irreducible representations (of differential rings). For GWA of an arbitrary degree some natural classes of simple modules are classified, namely, finite dimensional and strong D-torsion (sec. 5), polynomial (sec. 5), which are analogous of the weight and Whittaker's modules over simple Lie algebras, respectively.

In the spirit of I. Bernstein [5,7] the theory of the dimension and the multiplicity for finitely generated modules is developed (sec. 6–8). In contrast to Bernstein theory the multiplicity of a module is a fractional number, not necessarily an integer, and the growth of the modules is described not only by one (Hilbert) polynomial, but by several. Nevertheless, Bernstein's inequality is taking place for considered GWA (sec. 7). In sec. 9 the modules over the Weyl algebras A_n which have the (minimal possible Gelfand-Kirillov) dimension n and the multiplicity 1 are classified. They are the "smallest" simple A_n-modules.

Considered generalized Weyl algebras of degree n are the tensor product of n generalized Weyl algebras of degree 1 for which the simple modules are classified (for example, $A_n = A_1 \otimes \ldots \otimes A_1$, n times). Then the tensor product of simple modules over GWA of degree 1 is a simple module over the tensor product of algebras (sec. 10). In particular, $\hat{A}_1 \otimes \ldots \otimes \hat{A}_1 \subset \hat{A}_n, sl(2)^\wedge \otimes \ldots \otimes sl(2)^\wedge \subset (sl(2) \times \ldots \times sl(2))^\wedge$, (corollary 10.1), where $\hat{A}_1$ is the set of isomorphic classes of simple A_1-modules.

For considered generalized Weyl algebras the global homological dimension, the Krull and Gelfand-Kirillov dimension are computed (sec. 5), the criterion for simplicity of algebras is established (sec. 5), Stafford's Theorem [6] are generalized (sec. 13).

2 Definition of generalized Weyl algebras [1–3]

Let D be a ring, $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a set of commuting automorphisms of $D, (\sigma_i\sigma_j = \sigma_j\sigma_i$, for all $i, j), a = (a_1, \ldots, a_n)$ be a set of (non-zero) elements of the centre $Z(D)$ of D, such that $\sigma_i(a_j) = a_j$ for all $i \neq j$. The *generalized Weyl algebra* $A = D(\sigma, a)$ (briefly, GWA) of degree n with *basic* ring D is the ring generated by D and $2n$ indeterminates $X_1^+, \ldots, X_n^+, X_1^-, \ldots, X_n^-$, subject to the relations:

$$X_i^- X_i^+ = a_i, X_i^+ X_i^- = \sigma_i(a_i), X_i^\pm \alpha = \sigma_i^{\pm 1}(\alpha) X_i^\pm, \forall \alpha \in D\,,$$

$$[X_i^-, X_j^-] = [X_i^+, X_j^+] = [X_i^+, X_j^-] = 0\, \forall i \neq j\,,$$

where $[x, y] = xy - yx$. We say that a and σ are the sets of *defining* elements and automorphisms of A, respectively. If D is a Noetherian ring (resp., without zero divisors), then A is a Noetherian ring (resp., without zero divisors) [1,3].

The category of generalized Weyl algebras is closed under the tensor product (over a basic field) of algebras, under joining to the algebra a set $C = \{C_i | i \in I\}$ of independent commuting variables. An opposite algebra to GWA is also GWA. More precisely,

$$A \otimes A' = D \otimes D'(\sigma \cup \sigma', a \cup a'),$$

$$A^0 = D^0(\sigma^{-1}, \sigma(a)), \text{ where } \sigma^{-1} = (\sigma_i^{-1}), \sigma(a) = (\sigma_i(a_i)),$$

$$A[C] = D[C](\sigma, a).$$

In the latter case we can make a more complicated construction by putting an action of σ_i on C_i in many nontrivial ways.

For any vector $k = (k_1, \ldots, k_n) \in \mathbf{Z}^n$ we put $v_k = v_{k1}(1) \ldots v_{kn}(n)$, where for any $1 \leq i \leq n$ and $m \geq 0 v_{\pm m}(i) = (X_i^\pm)^m, v_0(i) = 1$. In the case where $n = 1$ we write v_m for $v_m(1)$. It follows from the definition of the GWA that

$$A = \bigoplus_{k \in \mathbf{Z}^n} A_k$$

is a $\mathbb{Z}^n$-graded algebra ($A_k A_e \subset A_{k+e}, \forall k, e \in \mathbb{Z}^n$), where $A_k = Dv_k$.

3 The principal object

Further, unless it is indicated otherwise, A is GWA of degree n with basic polynomial ring $D = K[H_1, \ldots, H_n]$ in n variables with coefficients in the algebraically closed field K of characteristic 0, $\sigma_i(H_j) = H_j - \delta_{ij}(1 \leq i, j \leq n,$ where $\delta_{ij} = 1,$ if $i = j$ and 0, otherwise). Every defining element $a_i \in K[H_i]$ of A is a polynomial of positive degree. GWA A is Noetherian without zero divisors, hence, by Goldie's theorem A has a quotient field.

The *Weyl algebra* $A_n = A_n(K)$ of degree n over a field K is an associative K-algebra generated by $2n$ indeterminates $X_1, \ldots, X_n, \partial_1, \ldots, \partial_n$, subject to the relations:

$$[X_i, X_j] = [\partial_i, \partial_j] = [\partial_i, X_j] = 0 \text{ for } i \neq j, [\partial_i, X_i] = 1\,.$$