# Tensor homological minimal algebras, global dimension of the tensor product of algebras and of generalized Weyl algebras

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**Abstract**

Let $d$ be the left homological dimension or weak dimension. In the present paper we give for a large class of algebras the answer to the question:

- when the dimension of the tensor product of algebras is the sum of dimensions of the multiples,

$$\mathrm{d}(\Lambda_1 \otimes \ldots \otimes \Lambda_n) = \mathrm{d}\Lambda_1 + \ldots + \mathrm{d}\Lambda_n.$$

Tensor $d$-minimal algebras are introduced and it is proved that a large class of algebras are tensor $d$-minimal algebras.

## 1 Introduction, main results

Denote by $Alg$ (respectively, $\mathcal{LN}$; $\mathcal{LFN}$; $\mathcal{SC}$) the class of all (respectively, left Noetherian; left Noetherian and finitely generated; somewhat commutative, see sec. 3) algebras over a fixed field $K$. In this paper the tensor product $\otimes$ means $\otimes_K$ and a module means a left module. Denote by d one of the following dimensions of rings: wd, the *weak* dimension; lgd, the *left homological* dimension; kd, the *Krull* dimension (in the sense of Gabriel-Rentschler [GR]).

In the present paper we give for a large class of (infinite dimensional non-commutative) algebras the answer to the question:

- when the dimension of the tensor product of algebras is the sum of dimensions of the multiples,

$$\mathrm{d}(\Lambda_1 \otimes \ldots \otimes \Lambda_n) = \mathrm{d}\Lambda_1 + \ldots + \mathrm{d}\Lambda_n. \tag{1.1}$$

Usually, (1.1) is not true. For example, if $L_n$ is the division ring of the polynomial ring $P_n = K[X_1, \ldots, X_n]$ in $n$ variables, then lgd $L_n \otimes L_n = n \neq 0 =$ lgd $L_n+$ lgd $L_n$ ([MR], Proposition 5.10, p. 251). Therefore, we should put on the algebras $\Lambda_i$ some restricted conditions for (1.1) to be satisfied. It is well known, that for all above dimensions d=wd, kd (respectively, lgd) ([Aus],[MR])

$$\mathrm{d}(A \otimes B) \geq \mathrm{d}A + \mathrm{d}B, \text{ for all (respectively, left Noetherian) algebras } A, B. \tag{1.2}$$

So it is natural to give the following

**Definition.** An algebra $A$ is a *tensor* d-*minimal* with respect to some class of algebras $\Omega$, if

$$\mathrm{d}(A \otimes B) = \mathrm{d}A + \mathrm{d}B, \text{ for each } B \in \Omega. \tag{1.3}$$

In the case wd (lgd) we say that the algebra $A$ is *tensor weak minimal* (*tensor homological minimal*), briefly, TWM (THM). The polynomial ring $P_n$ is tensor homological minimal with respect to *Alg*, since for each algebra $B$

$$\mathrm{lgd}(P_n \otimes B) = n + \mathrm{lgd}\, B = \mathrm{lgd}\, P_n + \mathrm{lgd}\, B \ (Hilbert's\, syzygy\, theorem).$$

The global dimension of the tensor product (over a commutative ring) of algebras was studied in [ERZ]. At the beginning of the article the authors write that "*the questions concerning the dimension of the tensor product of two algebras have turned out to be surprisingly difficult.*" Theorem 8 of [ERZ] states that the algebra $T_n(K)$ of all $n \times n$ matrices $(k_{ij})$ with entries in (commutative ring) $K$ satisfying $k_{ij} = 0$ for $i < j$ is THM and TWM with respect to *Alg*. In [Ha] the tensor homological (and weak) minimality was proved for matrix ring $M_n(K)$ with entries in (commutative ring) $K$. If the algebras $A$, $B$ and $A \otimes B$ are semi-primary and the global dimension gl.dim $A \otimes B$ $< \infty$, then gl.dim $A \otimes B$ = gl.dim $A+$ gl.dim $B$ ([Aus 1], Theorem 5).

We shall assume that all considered classes $\Omega$ contain the field $K$.

One of the aims of this paper is to give sufficient conditions for an algebra to be tensor d-minimal and, as a consequence, prove this property for many well-known algebras and their tensor products (for example, the Weyl algebra $A_n$ of degree $n$ over an algebraically closed resp. and uncountable field is THM with respect to $\mathcal{SC}$ resp. $\mathcal{LFN}$). Since many "good" algebras are the tensor product of tensor d-minimal ones (for example, the Weyl algebra $A_n = A_1 \otimes \ldots \otimes A_1, n$ times), so their dimension can be computed via (1.1), using Lemma 1.1 (which is an easy corollary of the definition above).

**Lemma 1.1** *Let $\Lambda_i$ $(i = 1, \ldots, n)$ be tensor d-minimal algebras with respect to $\Omega$ such that $\Lambda_i \otimes \Omega \subseteq \Omega$ for all $i$. Then the tensor product $\otimes_{i=1}^n \Lambda_i$ is a tensor d-minimal algebra with respect to $\Omega$ and d$(\otimes_{i=1}^n \Lambda_i \otimes B) = \sum_{i=1}^n \mathrm{d}\Lambda_i + \mathrm{d}B$ for each $B \in \Omega$.* ∎

The structure of the paper is as follows. In sec. 2 the definitions and examples of generalized Weyl algebras are presented; generalized Weyl algebras include many well-known algebras: the Weyl algebra $A_1$; the universal enveloping algebra $Usl(2)$ of the Lie algebra $sl(2)$; their quantum analogs and natural deformations $A_1(q), U_q sl(2), \Lambda(b)$; the quantum plane $\Lambda$; the quantum Heisenberg algebra $\mathcal{H}_q$.

For some of these algebras we prove their tensor d-minimality, where d=wd or lgd (see Corollary 1.5, below) and using (1.1), in many cases calculate their homological dimension.

In sec. 3 *filter-, finite-, factor-* and *subsplit* algebras and modules are introduced (for the purpose of convenience all the definitions are given at the begining of sec. 3) and their relations are examined.

If a ring $R$ is commutative, then wd $R$ = $\sup\{ \mathrm{wd}\, R_{\mathbf{m}} \mid \mathbf{m} \in$ Specm $R$, the set of maximal ideals of $R\}$, where $R_{\mathbf{m}}$ is the localization of $R$ at $m$. J.-E. Roos [R] generalized this result for some non-commutative rings. Namely, if for some ring $A$ there exist