ON THE TOPOLOGICAL HOCHSCHILD HOMOLOGY OF $\mathbf{Z}/p^k\mathbf{Z}$

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The topological Hochschild homology (THH) was introduced by M. Bökstedt [1]. It was proved in [9] that THH and MacLene homology [7] are isomorphic. Recently B.I. Dundas and R. McCarthy [4] proved that THH and stable K-theory in the sense of F.Waldhausen are isomorphic. Using topological methods Bökstedt has calculated $THH(\mathbf{Z})$ and $THH(\mathbf{Z}/p\mathbf{Z})$ [2]. The last result is closely related with L. Breen's calculation [3]. Recently V. Franjou, J. Lannes and L. Schwartz give algebraical calculations of $THH(\mathbf{Z}/p\mathbf{Z})$ and $THH(\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$ [5]. In this paper we give a similar calculation for $THH(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$. For another calculation of THH see [8].

The main result of this paper is the following

THEOREM. Let $k>1$. Then

$$\begin{aligned} dim\ THH_{2n}(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) &= 1+[\frac{n}{p}] \\ dim\ THH_{2n-1}(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) &= [\frac{n}{p}] \end{aligned}$$

Moreover the natural map

$$\alpha_k : THH_*(\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) \longrightarrow THH_*(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$$

is a monomorphism in each dimension and for any $m>k$ the restriction of the natural map

$$\beta_{mk} : THH_*(\mathbf{Z}/p^m\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) \longrightarrow THH_*(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$$

on the $Im(\alpha_m)$ is an isomorphism and is zero on the complement of $Im(\alpha_m)$.

Remark. Thus $THH(\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) \longrightarrow\ lim\ THH(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$ is an isomorphism.

Proof. We prefer to work with MacLane cohomology. Then theorem will follow from duality and [9]. For any small category $\mathbf{C}$ and ring R we denote the category of functors from $\mathbf{C}$ to R-mod by $\mathbf{C}$-R-mod. By Theorem A of [6] one has an isomorphism

$$H^*_{ML}(S,S) = Ext^*_{\mathbf{C}-S}(I,I)$$

where $\mathbf{C}$ is a full subcategory of the category S-mod containing all $S^n, n \geq 0$ and functor I is the inclusion $\mathbf{C} \longrightarrow S$-mod. For any functor

$$U : \mathbf{B} \longrightarrow \mathbf{A}$$

we write $U^* : \mathbf{A}$-S-mod$\longrightarrow \mathbf{B}$-$S$-mod for the induced functor. If U has a left adjoint F, then for each $X : \mathbf{A} \longrightarrow S$-mod and $Y : \mathbf{B} \longrightarrow S$-mod one has an obvious isomorphism:

$$Ext^*_{\mathbf{B}-S}(U^*X, Y) \cong Ext^*_{\mathbf{A}-S}(X, F^*Y).$$

As a particular case we obtain for any homomorphism of rings $f : R \longrightarrow S$ a natural isomorphism (compare also with 9.2 of [5])

$$H^*_{ML}(R, S) \cong Ext^*_{\mathbf{C}-R}(f^*I, f^*I)$$

where $f^* : \mathbf{C}$-S-$mod \longrightarrow \mathbf{C}$-$R$-$mod$ denotes the functor induced by f. It follows from the Grothendieck spectral sequence for composite functors, that for any functors

$$F : \mathbf{C} \longrightarrow S - mod,\ E : \mathbf{C} \longrightarrow R - mod$$

there exist a spectral sequence

$$E_2^{pq} = Ext^p_{\mathbf{C}-S}(F, Ext^q_R(S, E(-))) \Longrightarrow Ext^{p+q}_{\mathbf{C}-R}(f^*F, E).$$

Moreover the spectral sequence is compatible with natural left $Ext^*_{\mathbf{C}-R}(E, E)$-module and right $Ext^*_{\mathbf{C}-S}(F, F)$-modele structures.

Now we take $E = f^*F$ and $F = I$. It follows from the above isomorphism that $H^*_{ML}(R, S)$ has natural ring structure and there is a spectral sequence

$$E_2^{pq} = H^p_{ML}(S, Ext^q_R(S, S)) \Longrightarrow H^{p+q}_{ML}(R, S)$$

which is compatible with the left $H^*_{ML}(R, S)$-module and right $H^*_{ML}(S, S)$-module structures. We'll consider the case when $S = \mathbf{Z}/p\mathbf{Z}$ and $R = \mathbf{Z}$ or $\mathbf{Z}/p^k\mathbf{Z}$. The corresponding spectral sequences will be writen as $E^{**}_*(\mathbf{Z})$ and $E^{**}_*(\mathbf{Z}/p^k\mathbf{Z})$. In both cases one has an isomorphism of right $H^*_{ML}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$-modules:

$$(1) \qquad E_2^{*q}(R) = Ext^q_R(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z}) \otimes H^*_{ML}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$$