The Convergence of Krylov Subspace Methods for Large Unsymmetric Linear Systems*

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Abstract

The convergence problem of Krylov subspace methods, e.g. FOM, GMRES, GCR and many others, for solving large unsymmetric linear systems has been intensively investigated. There are many results in the literature, mainly for the case where the coefficient matrix A is diagonalizable and its spectrum lies in the open right (left) half plane. In this paper, we focus on a convergence analysis of those Krylov subspace methods which give rise to a similar minimax problem in the case where the coefficient matrix A is defective. When the spectrum of A either lies in the open right (left) half plane or is on the real axis, we establish the related theoretical error bounds and reveal some intrinsic relationships between the convergence speed and the spectrum of A. The results show that these methods are likely to converge slowly whenever one of three cases occurs: A is defective, the distribution of its spectrum is not favorable, or the Jordan basis of A is ill-conditioned.

Keywords. unsymmetric linear systems, Krylov subspace, the Chebyshev polynomials, defective, derivatives.

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1 Introduction

In recent years, considerable efforts have been devoted to solving large unsymmetric linear system

$$Ax = b, \tag{1}$$

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where $A \in C^{N\times N}$ is large, unsymmetric (non-Hermitian) and nonsingular, and b an $N\times 1$ column vector. A major class of methods for solving (1) is Krylov subspace type or conjugate gradient type methods, and so far many researchers have made contributions to them [2, 9, 8, 6, 11, 22, 23, 24, 25, 31, 32]; see the survey papers [26, 12]. There are some very successful schemes among them, e.g. GCR or ORTHOMIN [8, 9, 31], FOM or Arnoldi's method [22], GMRES [25] and QMR [11], to name only a few.

We can observe that as far as the convergence analysis is concerned, many of the Krylov subspace methods including those named above and some others, e.g. the method presented in [6], all give rise to a similar minimax problem. The only difference is that some of the methods require that the spectrum of the coefficient matrix A is restricted to the open right (left) half plane, e.g. GCR, while some of them do not, e.g. FOM and GMRES. Until now, several authors, e.g. Manteuffel [18, 19], Eisenstat et al. [8], Saad et al. [22, 25] and Freund et al. [11], have derived a number of results, mainly when A is diagonalizable and its spectrum lies in the open right (left) half plane.

This paper focuses on the case where A is defective and its spectrum either lies in the open right (left) half plane or is on the real axis, and studies the associated convergence problem of these methods. We establish the related theoretical error bounds and reveal some intrinsic relationships between the convergence speed and the spectrum of A. Here I want to point out that Trefethen [28, 20] has introduced some concepts of *ϵ-pseudo-spectra* to deal with non-normal matrices, which are very useful in making a convergence analysis. Indeed, in the presence of round errors, an original defective matrix will become a diagonalizable one with simple eigenvalues. However, the eigenvector matrix will be very ill-conditioned in that case. Numerical experiments have shown that one could not see the difference in convergence behaviour for matrices with simple eigenvalues, but ill-conditioned eigenvectors for some almost multiple eigenvalues, and matrices with those almost multiple eigenvalues replaced by Jordan blocks of appropriate size, e.g. [30]. Therefore, from both mathematical and numerical points of view, it is worthwhile to consider the Jordan block case in a convergence analysis, and our analysis may help to understand phenomena that are observed in actual floating-point computation.

In Section 2 we introduce the notation used; in Section 3, we briefly review FOM and GMRES from a theoretical point of view; taking FOM and GMRES as examples, we devote Section 4 to a convergence analysis of these methods.

2 Notation

This part is mainly taken from [13, 19].

Assume that $A \in C^{N\times N}, N \gg 1$, is unsymmetric (non-Hermitian). Let Q_m denote the set of all polynomials of degree not exceeding m and the Krylov subspace $\mathcal{K}_m(r^{(0)}, A) = \mathrm{span}\{r^{(0)}, Ar^{(0)}, \ldots, A^{m-1}r^{(0)}\}$, and let π_m be the orthogonal projector on $\mathcal{K}_m(r^{(0)}, A)$. Let $\|\cdot\|$ denote the 2-norm, the superscript "H" the conjugate