# On IGMRES($q$): Incomplete Generalized Minimal Residual Methods for Large Unsymmetric Linear Systems*

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**Abstract**

In this paper, we analyze in detail incomplete generalized minimal residual methods (IGMRES($q$)), proposed by Brown et al., for solving large unsymmetric linear systems, which are truncated versions of the generalized minimal residual method (GMRES), and establish a convergence theory of the methods. We derive the relationship between the residual norms for IGMRES($q$) and GMRES, and expose that IGMRES($q$) can work well and behave much like GMRES under the assumption that the basis vectors of Krylov subspace generated by IGMRES($q$) are strongly linearly independent. Also, we establish some relationships between the residual norms for IOMs($q$) and IGMRES($q$). Theoretical analysis shows that IGMRES($q$) may not behave like GMRES since the basis vectors generated by IGMRES($q$) can be nearly linearly dependent for any unsymmetric matrix and the parameter $q$. As for practical considerations, we suggest one strategy of choosing $q$. Finally, some numerical experiments are reported to show convergence behaviors of IGMRES($q$) and their restarted versions.

**Key words.** unsymmetric, Krylov subspace, IGMRES($q$), IOMs($q$), GMRES, convergence, restarted

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# 1 Introduction

Consider the solution of linear system

$$Ax = b, \tag{1}$$

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where $A \in C^{N \times N}$ is nonsingular, large and unsymmetric (non-Hermitian), and $b$ an $N \times 1$ column vector. A major class of methods for solving (1) is Krylov subspace or conjugate gradient type methods. So far, much work has been done; see, e.g. [12, 2, 3, 4, 41, 31, 32, 33, 35, 8, 9, 13, 18, 19, 29, 24, 25, 27] and many others. For survey papers, refer to [16, 36, 17, 14]; for a numerically comparative study of some of them, see [37, 38]; for a synthesis and a general description of available conjugate gradient type methods, see [34].

Of these methods, one kind of the most successful schemes is based on the orthogonal projection process, typified by GCR or ORTHOMIN [39, 41, 9], FOM or Arnoldi's method [31] and GMRES [35]. There are rich references concerning these methods, as mentioned previously. Here we want to stress that there is an intimate connection between FOM and GMRES. Brown [7] has shown that a singular upper Hessenberg matrix $H_m$ in FOM corresponds to the "stagnation" of approximate solutions in GMRES, while the occurrence of the "stagnation" of approximate solutions in GMRES is due to a singular upper Hessenberg matrix $H_m$. If successful, GMRES performs better than FOM since the residual norms of the former are characterized by a minimization property over the Krylov subspace generated by $A$. In contrast to those algorithms based on the Lanczos like process [26, 12, 13], GCR, FOM and GMRES use long recurrences, which means that work and storage per step grow as the number of steps increases. As a consequence, one has to restart these algorithms in practice, which may result in very slow convergence for some problems. Except for those matrices of very special forms, however, several researchers have shown that long recurrences cannot be simplified [10, 11, 23].

Even so, noticing the disadvantage of long recurrences, some researchers have proposed truncated strategies, which use only a few rather than all the previously computed vectors in recurrences and can thus be significantly less expensive. For example, Vinsome [39] suggested the truncated version ORTHOMIN($q$) of ORTHOMIN or GCR, Young et al. [41] presented the corresponding truncated versions ORTHODIR($q$) and ORTHORES($q$) of ORTHODIR and ORTHORES, Saad [31] proposed the truncated version IOMs($q$) of FOM, and Brown et al. [5, 6] suggested the truncated version IGMRES($q$) of GMRES for solving (1). At about the same time, Saad applied IOMs($q$) for solving eigenvalue problems [30]. Later, Saad [32, 33, 35], Brown et al. [5, 6] and Gear et al. [15] discussed and used IOMs($q$), and a number of researchers described or mentioned them, e.g. [8, 17, 27]. Moreover, Saad [33] conjectured that IOMs($q$) should converge in general provided that the parameter $q$ is sufficiently large. Besides, there are some relationships between various truncated versions, for example, Saad [33] has shown that IOMs($q$) are equivalent to ORTHORES($q$) in the mathematical sense.

A convergence analysis of the truncated ORTHOMIN($q$) has been made by [8], and a convergence theory of IOMs($q$) has been established by [21]. For all the other truncated versions, however, no convergence theory has been given.

We devote this paper to the study of IGMRES($q$), and establish a convergence theory.