A Sufficient Condition for every Class of Inverse Z–Matrices

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Abstract

Recently, a classification of matrices of class Z was introduced by Fiedler and Markham. This classification contains the classes of M–matrices and the classes of N_0– and F_0–matrices studied by K. Fan, G. Johnson, and R. Smith. The problem of determining which nonsingular matrices have inverses which are Z–matrices is called the *inverse Z–matrix problem.* For special classes of Z–matrices, such as the M–matrices and N_0–matrices, there exist at least partial results, i.e., special classes of matrices were introduced for which the inverse of such a matrix is an M–matrix or an N_0–matrix. Here, we define a system of classes of matrices for which the inverse of each matrix of each class belongs to one class of the classification of Z–matrices defined by Fiedler and Markham. Moreover, certain properties of the matrices of each class are established, e.g., inequalities for the sum of the entries of the inverse and the structure of certain Schur complements. We also give a necessary and sufficient condition for regularity. The class of inverse N_0–matrices given here generalizes the class of inverse N_0–matrices discussed by R. Smith. All results established here can be applied to a class of distance matrices which corresponds to a non-archimedean metric. This metric arises in the p-adic number theory and in taxonomy.

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1 Introduction and Preliminaries

Throughout this paper, we basically deal with $n \times n$ Z–matrices, i.e., real matrices whose off–diagonal elements are nonpositive. One well known class of Z–matrices is the class of nonsingular M–matrices, i.e., those matrices of the form $A = tI - B$ where B is a nonnegative matrix ($B \geq O$) and where $t > \rho(B)$; here, $\rho(B)$ denotes the spectral radius of B. However, other classes of Z–matrices are also of interest, and have been discused in the literature. Recently, Fiedler and Markham introduced in [FM] a classification of matrices in class Z. They defined the class L_s, for $s = 0, \ldots, n$, as consisting of real $n \times n$ Z–matrices which have the form

$$A = tI - B, \quad \text{where} \\ B \geq O \quad \text{and} \quad \rho_s(B) \leq t < \rho_{s+1}(B). \tag{1.1}$$

Here, $\rho_s(B)$ denotes the maximum of the spectral radii of all $s \times s$ principal submatrices of B, where $\rho_0(B) := -\infty$ and $\rho_{n+1}(B) := +\infty$. Thus, the class L_n is just the class of (singular and nonsingular) M–matrices. We note that L_{n-1} is the class of N_0–matrices introduced by G. Johnson [J2], and this class contains the N–matrices introduced by K. Fan [F]. Moreover, L_{n-2} is the class of F_0– matrices defined by R. Smith [S2].

As proved in [FM], for each s with $1 \leq s \leq n-1$, the class L_s is equal to the class of Z–matrices for which all principal submatrices of order s are M–matrices, but there exists a principal submatrix of order $s+1$ which is not an M–matrix. Additional properties of some of these classes are given in [BP], [S1] and [S3].

On the other hand, there has been interest in inverse M–matrices, i.e., any nonsingular matrix $B \geq O$ whose inverse is an M–matrix. A survey of this topic is given by C.R. Johnson [J1]. Recently, the new class of generalized ultrametric matrices was simultaneously introduced by McDonald, Neumann, Schneider, and Tsatsomeros [MNST] and Nabben and Varga [NV2]. If nonsingular, these matrices are inverse M–matrices. However, the oldest class of symmetric inverse M–matrices is the class of positive type D matrices defined by Markham [Mar] in 1972. A matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is of *type D* if there exist real numbers $\{\alpha_i\}_{i=1}^n$, with $\alpha_n > \alpha_{n-1} > \ldots > \alpha_1$, such that

$$a_{i,j} = \begin{cases} \alpha_i & if\ i \leq j; \\ \alpha_j & if\ i > j. \end{cases} \tag{1.2}$$

As shown in [Mar], a matrix of positive type D (i.e., with $\alpha_1 > 0$) is nonsingular, and its inverse is a symmetric M-matrix (i.e., a Stieltjes matrix). As an example, for $\alpha_j := j$ for $1 \leq j \leq n$, where $n \geq 2$, the associated $n \times n$ positive type D matrix and its inverse are