On IOMs(q): Incomplete Orthogonalization Methods for Large Unsymmetric Linear Systems*

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Abstract

The truncated version, Incomplete Orthogonalization Methods (IOMs(q)), of Full Orthogonalization Methods (FOM), proposed by Saad has been used a long time for solving large unsymmetric linear systems. However, no convergence analysis has been made. In this paper, we analyze in detail IOMs(q) from a theoretical point of view, and establish a convergence theory of the methods. We derive a number of important results of how the unsymmetry of the matrix A to affect the basis vectors generated by IOMs(q), and establish the relationships between the residuals for IOMs(q) and the full orthogonalization method (FOM). The results show that IOMs(q) behave much like FOM once the basis vectors generated by IOMs(q) are well conditioned. However, it is shown that IOMs(q) may generate an ill-conditioned basis for any unsymmetric matrix and the parameter q such that IOMs(q) may fail to converge or at least cannot behave like FOM. Due to the mathematical equivalence between IOMs(q) and the truncated ORTHORES(q) developed by Young and Jea, essentially we give insights into the convergence of ORTHORES(q). As for practical considerations, we suggest one strategy of choosing the parameter q. Numerical experiments are reported to show convergence behaviors of IOMs(q).

Key words. unsymmetric, Krylov subspace, IOMs(q), FOM, Arnoldi, truncated, basis vector, convergence, restarted

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1 Introduction

Consider the solution of linear system

$$Ax = b, \tag{1}$$

where $A \in C^{N\times N}$ is nonsingular, large and unsymmetric (non-Hermitian), and b an $N \times 1$ column vector. A major class of methods for solving (1) is Krylov subspace or conjugate gradient type methods. So far, much work has been done; see, e.g. [12, 2, 3, 4, 42, 31, 32, 33, 35, 8, 9, 13, 19, 20, 29, 24, 25, 27] and many others. For survey papers, refer to [17, 36, 18, 14]; for a numerically comparative study of some of them, see [37, 38]; for a synthesis and general description of available conjugate gradient type methods, see [34].

Of these methods, one kind of the most successful schemes is based on the orthogonal projection, typified by GCR or ORTHOMIN [39, 42, 9], ORTHODIR and ORTHORES [42], FOM or Arnoldi's method [31] and GMRES [35]. There are rich references concerning these methods, as mentioned previously. Here we want to stress that there is an intimate connection between FOM and GMRES. Brown [7] has shown that a singular upper Hessenberg matrix H_m in FOM corresponds to the "stagnation" of approximate solutions in GMRES, while the occurrence of the "stagnation" of approximate solutions in GMRES is due to a singular upper Hessenberg matrix H_m. If successful, GMRES performs better than FOM since the residual norms of the former are characterized by a minimization property over Krylov subspace generated by A. In contrast to those algorithms based on the Lanczos like process [26, 12, 13], GCR, ORTHODIR, ORTHORES, FOM and GMRES use long recurrences, which means that work and storage per step grow as the number of steps increases. As a consequence, one has to restart these algorithms in practice, which may result in very slow convergence for some problems. Except for those matrices of very special forms, however, several authors have shown that long recurrences cannot be simplified [10, 11, 23].

Even so, noticing the disadvantage of long recurrences, some researchers have proposed truncated strategies, which use only a few rather than all previously computed vectors in recurrences and can thus be significantly less expensive. For example, Vinsome [39] suggested the truncated version ORTHOMIN(q) of ORTHOMIN or GCR, Young et al. [42] presented the corresponding truncated versions ORTHODIR(q) and ORTHORES(q) of ORTHODIR and ORTHORES, Saad [31] proposed the truncated version IOMs(q) of FOM, and Brown et al. [5, 6] suggested the truncated version IGMRES(q) of GMRES for solving (1). At about the same time, Saad applied IOMs(q) for solving unsymmetric eigenvalue problems [30]. Later, Saad [32, 33, 35], Brown et al. [5, 6] and Gear et al. [15] discussed and used IOMs(q), and some others described or mentioned IOMs(q), e.g. [8, 18, 27]. Besides, there are some relationships between various truncated versions, for example, Saad [33] has shown that IOMs(q) are equivalent to ORTHORES(q) in the mathematical sense.