First Sincere Module

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1 Introduction

Throughout this note all algebras are assumed to be connect, basic finite-dimensional (associative) algebras with 1 over an algebraically closed field k and all modules are finite generated left modules. A preprojective component $\mathcal{P}$ of the Auslander-Reiten quiver of an algebra A is called a sincere preprojective component if there is a sincere indecomposable module M in $\mathcal{P}$. The main purpose of this note is to show the following

Theorem *Let $\mathcal{P}$ be a sincere preprojective component of the Auslander-Reiten quiver of an algebra A. Assume that the orbit graph of $\mathcal{P}$ is a tree. Then there exists unique sincere indecomposable module M in $\mathcal{P}$, such that $Hom_A(M, X) \neq 0$ for all sincere indecomposable A-module X.*

Note that in case A is a tree algebra the result was proved already by Unger in [**U**]. Thus the above result is a generalization of the main result in [**U**]. The module M determined by the theorem above is preprojective. Moreover all preprojective, sincere modules are successors of M. For this reason we call M the first sincere module. We will use the terminology of

[**R**]. We say that an indecomposable module V is a predecessor of a slice $\mathcal{S}$ if there exist some $[X] \in \mathcal{S}$ such that V is a predecessor of X. Let $\mathcal{P}$ be a sincere preprojective component of the Auslander-Reiten quiver of an algebra A. We denote by Ψ the module class consisting of all sincere indecomposable modules in $\mathcal{P}$. For an indecomposable module V in $\mathcal{P}$ and a slice $\mathcal{S}$ in $\mathcal{P}$, we denote by $\mathcal{S} \preceq V$ if V is a successor of some indecomposable module which lays on $\mathcal{S}$, and denote by $\mathcal{S} \preceq \Psi$ if $\mathcal{S} \preceq X$ for each $X \in \Phi$. For the slice $\mathcal{S}$, we put $\tau\mathcal{S}$=$\{$ $\tau[X] \mid [X] \in \mathcal{S}\}$ where τ is the Auslander-Reiten translate.

The following result is a direct conseqence of the theorem since the orbit graph of a directed sincere algebra is a tree [**BLS**].

Corollary *Let A be a sincere directed algebra. Then there exists a unique sincere indecomposable module M such that $Hom_A(M, X) \neq 0$ for all sincere indecomposable modules X, and there exists a unique sincere indecomposable module N such that $Hom_A(X, N) \neq 0$ for all sincere indecomposable modules X.*

2 Proof of Theorem

Lemma 1 *Let $\mathcal{P}$ be a preprojective component of the Auslander-Reiten quiver of an algebra A, P a indecomposable projective module, and $\mathcal{S}$ a slice on $\mathcal{P}$. Then*

$$\{[X] \in \mathcal{S} | Hom_A(P, X) \neq 0\}$$

is connected.

Proof. Because $\mathcal{S}$ is a slice, by tilting theory, there is a hereditary algebra C and a tilting module ${}_CT$ with $\mathrm{End}_C(T) = A$ and

$$\{\mathrm{Hom}_C({}_CT, I) \mid I \textit{ is an indecomposable injective module}\} = \{[X] \in \mathcal{S}\}.$$

The functor $F = \mathrm{Hom}_C({}_cT, -)$ gives an equivalence between the full subcategory $\mathcal{G}$ of $C-$mod with objects generated by ${}_CT$ and the full subcategory