Double of the Yangian and rational R-matrices

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Abstract

Studying the algebraic structure of the double $\mathcal{D}Y(g)$ of the yangian $Y(g)$ we present the triangular decomposition of $\mathcal{D}Y(g)$ and a factorization for the canonical pairing of the yangian with its dual inside $\mathcal{D}Y(g)$. As a consequence we obtain an explicit formula for the universal R-matrix of $\mathcal{D}Y(g)$ and demonstrate how it works in evaluation representations of $Y(sl_2)$. We interprete one-dimensional factor arising in concrete representations of R as bilinear form on highest weight polynomials of irreducible representations of $Y(g)$ and express this form in terms of Γ functions.

1 Introduction

Yangian $Y(g)$ of a simple Lie algebra g was introduced by V.Drinfeld [D1] as a deformation of universal enveloping algebra $U(g[t])$ of a current algebra $g[t]$. The yangians $Y(g)$ and quantum affine algebras $U_q(\hat{g})$ play the role of dynamical symmetries in quantum field theories [BL] [S]. Tensor products of finite-dimensional representations of yangians produce rational solutions of the Yang-Baxter equation; tensor products of finite-dimensional representations of quantum affine algebras produce trigonometric solutions of the Yang-Baxter equation. One can find out other deep parallels in representation theories of yangians and of quantum affine algebras. Nevertheless both of them have their own original features. Yangian $Y(g)$ is much more closer to classical Lie algebras, at least it contains universal enveloping algebra $U(g)$ as a subalgebra; moreover the yangians $Y(gl_n)$ could be defined entirely in terms of classical representation theory [O]. The structure of

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quantum affine $U_q(\hat{g})$ is more complicated. On the other hand, $U_q(\hat{g})$ inhabit the main properties of contragredient algebras; Chevalley generators and q-deformed Serre relations are permanent participants of the games with quantum affine algebras.

The theory of Cartan-Weyl basis [KT2] [KT3] allows to describe explicitely the universal R-matrix, one of the main object in physical applications. We have nothing of this for the yangians, though suitable modifications of classical methods of representation theory work for $Y(g)$ and $U_q(\hat{g})$ as well (see [Ch] [D1] [J] [NT] for instance). We want to make this gap smaller.

It is more reasonable to work with quantum double $\mathcal{D}Y(g)$ of the yangian, if we take in mind physical applications. The representations of $\mathcal{D}Y(g)$ do not differ much from the representations of yangian $Y(g)$: the extension of a representatin of $Y(g)$ to representation of $\mathcal{D}Y(g)$ can be achieved just by reexpansion of the currents from $Y(g)$ in other point of projective line. We present here a study of some algebraic properties of $\mathcal{D}Y(g)$, with the accent to the canonical pairing in the double.

We prove that $\mathcal{D}Y(g)$ itself and a Hopf pairing of $Y(g)$ with its dual inside $\mathcal{D}Y(g)$ admit triangular decomposition analogous to Gauss decomposition of ordinary matrices. This property gives possibility to describe the pairing quite explicitely. As a consequency we obtain explicite factorized expression for the universal R-matrix of $\mathcal{D}Y(g)$ (completely proved for $\mathcal{D}Y(sl_2)$ and partially in general case). To make the formulas more transparant, we present detailed calculations for $\mathcal{D}Y(sl_2)$, including the action of the universal R-matrix on evaluation representations.

The most interesting factor R_H of the universal R-matrix is concerned with (zero charge) Heisenberg subalgebra of $\mathcal{D}Y(g)$ which is a deformation of the currents to Cartan subalgebra h of g. Analogously to the case of $U_q(\hat{g})$ [KT2] [KST] the structure of R_H is governed by the q-analog of invariant scalar product in h; whenever R_H acts on representations of $\mathcal{D}Y(g)$ a variable q becomes a shift operator $T: \; Tf(x) = f(x+1)$ (for quantum affine algebras q goes to multiplicative shift $T_q f(x) = f(qx)$ in analogous situation).

After substitution of the universal R-matrix into tensor product of concrete representations of $\mathcal{D}Y(g)$ we obtain more then usual rational R-matrix; an additional information is concentrated in scalar phase factor (scalar S-matrix) which we interprete as bilinear multiplicative form on highest weight polynomials of finite-dimensional representations V of $Y(g)$ (or, equivalently, on $K_0(RepY(g))$. This form is a deformation of skewsymmetric form $\frac{<,>}{a-b}$ on irreducible evaluation representations of $g[t]$ where $<,>$ is invariant scalar product in h^*; a and b are the points where evaluation representations are living. We present an explicit expression of this form as some ratio of Γ functions defined by the structure of q-analog of invariant scalar product in h.

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