Some Hall Polynomials for Representation-finite Trivial Extension Algebras

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Let k be a finite field and assume that Λ is a finite dimensional associative k-algebra with 1. Denote by modΛ the category of all finitely generated (right) Λ-modules and by indΛ the full subcategory in which every object is a representative of the isoclass of an indecomposable (right) Λ-module. We are interested in the existance of the Hall polynomial φ^M_{NL} for any $L, M, N \in$modΛ (for the definition, see [**7,8**] or Section 1 below). In case Λ is directed, C. M. Ringel in [**7**] has shown that Λ has Hall polynomials, and in case Λ is cyclic serial, the same result has also been obtained by J. Guo [**4**]. It has been conjectured in [**8**] that any representation-finite k-algebra has Hall polynomials. In this investigation, we shall show that if Λ is a representation-finite trivial extension algebra, then, for any $L, M, N \in$modΛ with N indecomposable, Λ has the Hall polynomials φ^M_{LN} and φ^M_{NL}. Using these Hall polynomials, we can naturally structure the free abelian group with a basis indΛ, denoted by $K($mod$\Lambda)$, into a Lie algebra and the universal enveloping algebra of $K($mod$\Lambda) \otimes_{\mathbf{Z}} \mathbf{Q}$ is just $\mathcal{H}(\Lambda)_1 \otimes_{\mathbf{Z}} \mathbf{Q}$, where $\mathcal{H}(\Lambda)_1$ is the degenerated Hall algebra of Λ (see Section 5 below).

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1 Preliminaries

For M, $N_1, \ldots, N_t \in$modΛ, we denote by $F^M_{N_t \ldots N_1}$ the number of filtrations

$$0 = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_t = M$$

of M such that $U_i/U_{i-1} \simeq N_i$ for all $1 \leq i \leq t$. In particular, F^M_{LN} is the number of the submodules U of M such that $U \simeq N$ and $M/U \simeq L$.

Let E be a field extension of k. For any k-space V, we denote by V^E the E-space $V \otimes_k E$. Of course, Λ^E naturally becomes a E-algebra. We recall from [**8**] that E is conservative for Λ if for any indecomposable $M \in$modΛ, $(\mathrm{End}M/\mathrm{radEnd}M)^E$ is a field. Under field isomorphisms we put

$$\Omega = \{ \text{ a finite field extension } E \text{ of } k \mid E \text{ is conservative for } \Lambda\}.$$

Note that if Λ is representation-finite, then Ω is an infinite set.

Proposition 1.1 *Let E be a field conservative extension of k. Denote by $(mod\Lambda)^E$ the full subcategory of $mod\Lambda$ which objects are M^E for all $M \in mod\Lambda$. Then*

i). $M \in mod\Lambda$ is indecomposable if and only if $M^E \in mod\Lambda^E$ is indecomposable. In particular, M is an indecomposable projective (respectively, injective) Λ-module if and only if M^E is an indecomposable projective (respectively, injective) Λ^E-module; and for any $M \in mod\Lambda$, $(radM)^E = radM^E$ and $(socM)^E = socM^E$.

ii). If $M \xrightarrow{f} N$ is a source (respectively, a sink) map in $mod\Lambda$, then $M^E \xrightarrow{f\otimes 1_E} N^E$ is also a source (respectively, a sink) map in $mod\Lambda^E$.

In particular, if Λ is representation-finite, then $(mod\Lambda)^E = mod\Lambda^E$.

Proof. Since E is conservative, we have $(\mathrm{radEnd}_\Lambda M)^E = \mathrm{radEnd}_{\Lambda^E} M^E$ for any $M \in \mathrm{ind}\Lambda$.

i). From the definition of E, it is clear that $M \in \mathrm{mod}\Lambda$ is indecomposable if and only if $M^E \in \mathrm{mod}\Lambda^E$ is indecomposable. In particular, M is an indecomposable projective Λ-module if and only if M^E is an indecomposable projective Λ^E-module. Therefore given a complete set of orthogenal primary idempotents $\{e_1, e_2, \ldots, e_n\}$ of Λ, we obtain a complete set of orthogenal primary idempotents $\{e_1 \otimes 1_E, e_2 \otimes 1_E, \ldots, e_n \otimes 1_E\}$ of Λ^E and

$$e_i \otimes 1_E(\mathrm{rad}\Lambda)^E e_i \otimes 1_E = \mathrm{rad}(e_i \otimes 1_E \Lambda^E e_i \otimes 1_E) = (\mathrm{rad}(e_i\Lambda e_i))^E = (e_i \mathrm{rad}\Lambda e_i)^E$$

for $i = 1, 2, \ldots, n$ and hence

$$\begin{aligned} \mathrm{rad}\Lambda^E &= \textstyle\sum_{i=1}^n e_i \otimes 1_E \mathrm{rad}\Lambda^E e_i \otimes 1_E + \sum_{i\neq j} e_i \otimes 1_E \Lambda^E e_j \otimes 1_E \\ &= (\textstyle\sum_{i=1}^n e_i \mathrm{rad}\Lambda e_i + \sum_{i\neq j} e_i\Lambda e_j)^E \\ &= (\mathrm{rad}\Lambda)^E. \end{aligned}$$

And for any $M \in \mathrm{mod}\Lambda$,

$$\mathrm{rad}M^E = \mathrm{rad}\Lambda^E M^E = (\mathrm{rad}\Lambda)^E M^E = (\mathrm{rad}M)^E.$$

By duality, we can consider the injective and the socle.

ii). By i), it is enough to show that if

$$\alpha: \ 0 \longrightarrow N \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$$

is an Auslander-Reiten sequence in $\mathrm{mod}\Lambda$, then

$$\alpha^E: \ 0 \longrightarrow N^E \xrightarrow{f\otimes 1_E} L^E \xrightarrow{g\otimes 1_E} M^E \longrightarrow 0$$

is also an Auslander-Reiten sequence in $\mathrm{mod}\Lambda^E$. In fact, according to [**1**], we easily see that by the structure $N^E \simeq (DTrM)^E = DTrM^E$ and that α^E is not split; and, by the structure of Auslander-Reiten sequences, it is enough to see that α^E belongs to the (simple) socle of $\mathrm{End}M^E - \mathrm{End}N^E-$bimodule $\mathrm{Ext}^1_{\Lambda^E}(M^E, N^E)$, that is,

$$(\mathrm{radEnd}_{\Lambda^E} M^E)\alpha^E(\mathrm{radEnd}_{\Lambda^E} N^E) = 0.$$

But $(\mathrm{radEnd}M)\alpha(\mathrm{radEnd}N) = 0$. Therefore

$$(\mathrm{radEnd}_{\Lambda^E} M^E)\alpha^E(\mathrm{radEnd}_{\Lambda^E} N^E) = ((\mathrm{radEnd}_\Lambda M)\alpha(\mathrm{radEnd}_\Lambda N))^E = 0.$$