Lie Algebras determined by Finite Auslander-Reiten Quivers

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Let Λ be a finite-dimensional associative k- algebra with 1 over some field k and Γ the Auslander-Reiten quiver of Λ. Denote by Γ_0 the vertex set of Γ. Set $\mathcal{L}(\Gamma) = \oplus_{x\in\Gamma_0}\mathbf{Z}x$, the free abelian group generated by Γ_0. It is interesting to make $\mathcal{L}(\Gamma)$ into a Lie algebra using the representation theory of algebras. C.M.Ringel in [Rin1,2] showed that if there exist the Hall polynomials of Λ (or Γ), such as directed algebras, then $\mathcal{L}(\Gamma)$ has a Lie structure via Hall algebras over finite fields. On the other hand, Ch.Riedtmann in [**3**], for any representation-finite algebra over $\mathbf{C}$, also obtained a Lie structure of $\mathcal{L}(\Gamma)$ using the Euler-Poincaré characteristic of some locally closed algebraic varieties. In case Λ is a representation-finite hereditary with some Dynkin class, two kinds of the above Lie structures are coincident and $\mathbf{C}\otimes\mathcal{L}(\Gamma)$ is isomorphic to the positive part n_+ of a $\mathbf{Z}$-form of the complex semisimple Lie algebra with the same Dynkin class (see [**7**] and [**3**]). But in general, we don't know if any representation-finite algebra has Hall polynomials. C.M.Ringel conjectured in [**6**] this is true. Even if we have the existance of Hall polynomials, we don't know if such two Lie structures are coincident.

In this investigation, for any finite connected Auslander-Reiten quiver Γ with valuation 1, using the methods of C.M.Ringel in [**Rin1,3**] and Ch.Riedtmann in [**4**], we obtain a Lie structure of $\mathcal{L}(\Gamma)$ as a Lie subalgebra of $\mathcal{H}(\Gamma)_1$, where $\mathcal{H}(\Gamma)_1$ is the degenerate Hall algebra, see Section 1 below, and similarly with [**4**] we have $\mathcal{L}(\Gamma)\simeq\mathcal{L}(\tilde{\Gamma})/G$, the "orbit" Lie algebra of the universal cover $\tilde{\Gamma}$ of Γ (see [**4**] or Section 1 below), where $\mathcal{L}(\tilde{\Gamma})$ is the Lie algebra determined by $\tilde{\Gamma}$ via Hall algebras of $\tilde{\Gamma}$. If Γ has Hall polynomials, then $\mathcal{L}(\Gamma)$ has the Lie structure in the sense of Ringel [**6**]. In this case, as a consequence, we obtain $\mathcal{L}(\Gamma)=\mathcal{L}(\tilde{\Gamma})/G$

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1 Prelemenaries and Statement of Theorem

Throughout this paper, by an algebra Λ we meant a finite-dimensional algebra over some field k, which is associative and unitary, and by a Λ-module M we meant a finitely generated unitary Λ-module M.

Given a finite field k and a k-algebra Λ. We recall the integral Hall algebra $\mathcal{H}(\Lambda)$ of Λ from [**5**]. Let $N_1, N_2, \cdots, N_t$ and M be Λ-modules. We denote by $F^M_{N_1N_2\cdots N_t}$ the munber of the filtrations

$$M = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_t = 0$$

of M such that $U_{i-1}/U_i \simeq N_i$, for $1 \leq i \leq t$. Let $\mathcal{H}(\Lambda)$ be the free abelian group with basis $(u_{[M]})_{[M]}$, indexed by the set of isomorphism classes of Λ-modules. Instead of $u_{[M]}$ we will also write u_M. We define on $\mathcal{H}(\Lambda)$ a multiplication by the following rule:

$$u_{[N_1]}u_{[N_2]} = \sum_{[M]} F^M_{N_1N_2} u_{[M]};$$

note that on the right, we deal with a finite sum. In this way, $\mathcal{H}(\Lambda)$ become an associative ring with $1 = u_{[0]}$, and is called the integral Hall algebra of Λ.

Given Λ-modules N_1, N_2 and M. We denote by $\mathrm{Ext}^1_\Lambda(N_1, N_2)_M$ the subset of $\mathrm{Ext}^1_\Lambda(N_1, N_2)$ consisting of all exact sequences of form $0 \to N_2 \to M \to N_1 \to 0$. Then we have the following relation given in [**2**].

Proposition 1.1

$$F^M_{N_1N_2} = \frac{|Ext^1_\Lambda(N_1, N_2)_M||Aut_\Lambda M|}{|Aut_\Lambda N_1||Aut_\Lambda N_2||Hom_\Lambda(N_1, N_2)|}$$

Here, for any set N, we denote its cardinality by $|N|$

Throughout this paper, we shall assume that Γ is a translation quiver with trivial valuation (that is, valuation 1), which is finite and connected with vertex set Γ_0. Denote by $\mathcal{B}$ the set of all functions $\Gamma_0 \to \mathbf{N}_0$. By abuse of language, we denote the characteristic function for $x \in \Gamma_0$ just by x. For any $a, b \in \mathcal{B}$, we can define $a + b \in \mathcal{B}$ by $(a + b)(x) = a(x) + b(x)$ for any $x \in \Gamma_0$. Thus for any $a \in \mathcal{B}$ we have a standard decomposition $a = \sum_{x\in\Gamma_0} a(x)x$. Denote $\mu(a) = \sum_{x\in\Gamma_0} a(x)$.

Γ will be said to be an Auslander-Reiten quiver provided there exist a field k and a k-algebra Λ auch that $\Gamma = \Gamma_\Lambda$, where Γ_Λ is the Auslander-Reiten quiver of Λ. In this case, for any $x \in \Gamma_0$, we choose a representative $M(x) = M(\Lambda, x)$ of indecomposable Λ-modules corresponding to x, and for any $a \in \mathcal{B}$, we denote by $M(a) = M(\Lambda, a)$ the Λ-module $\oplus_{x\in\Gamma_0} a(x)M(x)$. Note that Γ is of trivial valuation. So in case Γ is an Auslander-Reiten quiver, then for any field k, there exists a unique standard k-algebra via the mesh category $k(\Gamma)$ of Γ, denoted by $\Lambda(k)$, such that $\Gamma = \Gamma_{\Lambda(k)}$. In this case, for any $x \in \Gamma_0$ and any $a \in \mathcal{B}$, we also denote $M(k, x) = M(\Lambda(k), x)$ and $M(k, a) = M(\Lambda(k), a)$.

In [**2**], we obtained the following result.

Proposition 1.2 *Let Γ be an Auslander-Reiten quiver. Then*

i). For any $a, b \in \mathcal{B}$, there exists an integer $h(a, b) \in \mathbf{N}_0$ such that

$$|Hom_{\Lambda(k)}(M(k, a), M(k, b))| = |k|^{h(a,b)}$$

for any finite field k.

ii). For any $a \in \mathcal{B}$, we have

$$|Aut_{\Lambda(k)}M(k, a)| = \prod_{x,y\in\Gamma_0,\ x\neq y} |k|^{h(x,y)a(x)a(y)} \prod_{x\in\Gamma_0} \prod_{i=1}^{a(x)} (|k|^{h(x,x)a(x)} - |k|^{(h(x,x)-1)a(x)+(i-1)})$$

for any finite field k.