Classifications and Characterisations of Snarks

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Abstract

This paper considers vertex-reductions of snarks. Each reduction naturally divides the class of snarks into three classes. The first class contains those snarks which are not reducible to a snark. The second contains those which are not reducible to a 3-colorable graph and the third one consists of the snarks which are reducible to a snark as well as to a 3-colorable graph. We characterise all these classes in terms of 2-factors.

Furthermore we characterise snarks which are divided into 3-colorable components by the removal of any edge cutset in terms of 3-critical subgraphs and show that there are infinitely many such snarks.

1 Introduction

We follow the terminology and notation of Bondy and Murty [2]. We consider proper edge colorings of 3-regular graphs. The set of 3-regular graphs is divided into two classes by a famous theorem of Vizing which states that each simple loopless graph is edge-colorable with Δ or $\Delta+1$ colors, see e.g. [2]. Class I contains the 3-colorable graphs and class II the graphs of chromatic index 4. Isaacs [12] called the graphs of class I *colorable* graphs and the graphs of class II *uncolorable* graphs. Uncolorable graphs are of great interest, since if there are counter-examples to some hard conjectures, e.g. Tutte's 5-flow conjecture, the double cover conjecture for 3-regular graphs (see e.g. [13]), then they must be uncolorable. Gardner [7] proposed to call nontrivial uncolorable graphs snarks. However it is not clear what is "nontrivial". The general opinion of a "nontrivial" snark seems to be that it is not reducible to a smaller snark by a well defined reduction. This topic has attracted considerable discussion in the literature and different types of reductions were investigated, see e. g. [4], [7], [12] and [15]. In none of these papers structural characterisations of the irreducible snarks are given.

As proposed in [16] and [17] another approach is to focus on the the structure of snarks, leaving the definition of a snark as broad as possible, and to find a recursive description of snarks.

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We pursue this approach and consider vertex-reductions of snarks. We define a *snark* to be a connected, bridgeless, simple, uncolorable graph.
A graph G is *vertex-reducible* if a simple graph G' can be obtained from G by removing two vertices together with all incident edges from G and adding an appropriate number of edges to restore 3-regularity. We also say G is vertex-reducible to G'. This reduction divides the class of snarks, denoted with $\mathcal{G}$, into three classes.

$\mathcal{B} := \{G | G \in \mathcal{G} \wedge G \text{ is not vertex-reducible to a snark}\}$

$\mathcal{S} := \{G | G \in \mathcal{G} \wedge G \text{ is not vertex-reducible to a colorable graph}\}$

$\mathcal{C} := \mathcal{G} \setminus (\mathcal{B} \cup \mathcal{S})$

The class $\mathcal{B}$ is the base, and the class $\mathcal{S}$ consists of strong snarks, that is snarks which are not vertex-reducible to a colorable graph. The class $\mathcal{C}$ consists of snarks which are vertex-reducible to a snark as well as to a colorable graph. One aim of this paper is to give a structural characterisation of these classes in terms of 2-factors.
We distinguish three kinds of vertex-reductions for the analysis of vertex-reductions of snarks. For $i = 1, 2$ we say G is (distance) *i-reducible* if it is vertex-reducible by removing two vertices v and w with $d_G(v, w) = i$. A graph G is (distance) *3-reducible* if it is vertex-reducible by removing two vertices v, w with $d_G(v, w) \geq 3$. 1-reductions are also called edge-reductions. The classes $\mathcal{B}_i$, $\mathcal{C}_i$ and $\mathcal{S}_i$ are defined as $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{S}$ with respect to i-reductions, for $i = 1, 2, 3$, and therefore

$$\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$$
$$\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3$$

Uncolorable graphs with bridge are of minor interest, since all conjectures mentioned above as well as the famous theorem of Tait (see e.g. [2]), which was the starting point for the study of snarks, are for bridgeless graphs. They are also not interesting from the view of reductions. It is easy to see that the only irreducible uncolorable graph with bridge can be obtained from two K_4 by inserting a vertex in an edge of each K_4 and joining these two vertices by an edge. All the other graphs with bridges are edge-reducible to a smaller graph with bridge, see [1].
It is natural to consider edge-reductions of snarks since each bridgeless (3-regular) graph $G \neq K_4$ can be reduced to a smaller bridgeless 3-regular graph by removing two adjacent vertices or by removing a square with a diagonal edge and restoring 3-regularity, see [1]. Edge-reductions are also directly related to Tutte's conjecture that every snark contains a subgraph contractable to the Petersen graph, see e.g. [17].
In section 2 we introduce the *color number* $c(G)$ of a graph G, the minimal cardinality of a color class of G. We show, for each even $n \geq 10$ there is a snark G with $c(G) = 2$. This invariant of a graph is a useful tool for the proofs of the characterisation theorems which are given in the sections 3, 4 and 5. In section 6 we investigate snarks of the bases. We show that the elements of $\mathcal{B}$ are at least cyclically 5-edge connected, and that $\mathcal{B}_1$ is an infinite set. In particular we show that each hypohamiltonian snark is an element of $\mathcal{B}_1$.