DECOMPOSITIONS AND REDUCTIONS OF SNARKS

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Abstract. According to M. Gardner [9], a snark is a non-trivial cubic graph whose edges cannot be properly coloured by three colours. The problem of what 'non-trivial' means is implicitly or explicitly present in most papers on snarks, and is the main motivation of the present paper. Our approach to the discussion is based on the following observation. If G is a snark with a k-edge-cut producing components G_1 and G_2, then either one of G_1 and G_2 is not 3-edge-colourable, or by adding a 'small' number of vertices to either component one can obtain snarks $\tilde{G}_1$ and $\tilde{G}_2$ whose order does not exceed that of G. The two situations lead to a definition of a k-reduction and k-decomposition of G. Snarks that for $m < k$ do not admit m-reductions, m-decompositions or both are k-irreducible, k-indecomposable and k-simple, respectively. The irreducibility, indecomposability and simplicity provide natural measures of non-triviality of snarks closely related to cyclic connectivity. The present paper is devoted to a detailed investigation of these invariants. The results give a complete characterization of irreducible snarks and characterizations of k-simple snarks for $k \leq 6$. A number of problems that have arisen in this research conclude the paper.

1. Introduction

Non-trivial cubic graphs whose edges cannot be properly coloured with three colours have been investigated for more than 100 years. The primary incentive for their study was the observation made by P. G. Tait [23] that the four-colour theorem is equivalent to the statement that every bridgeless planar cubic graph is 3-edge-colourable.

Since the four-colour theorem is true, we know that no non-trivial planar uncolourable cubic graphs exist. However, even non-planar graphs of this kind, such as the Petersen graph (shown in Fig.1), have been extremely difficult to find. In fact, until 1975 when Isaacs constructed two infinite families of them only four such graphs had been known. This led Martin Gardner [9] to "*propose calling nontrivial uncolorable trivalent graphs Snarks*", borrowing their name from the famous nonsense ballad "The Hunting of the Snark" by Lewis Carroll.

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The ambiguous word 'non-trivial' in this definition has attracted considerable discussion in the literature (see, e.g., [2, 4, 13, 21, 24, 25]). At any rate, 'non-trivial' means primarily a graph without a bridge. Besides having a bridge, however, a snark may possess some other features that make it 'trivial'; for instance if it is but a 'trivial' variation of another snark. It has been customary to consider trivial any snark that has a digon or a triangle because these are trivial features that can be added to or removed from a snark without altering its uncolourable property (see Fig.2 and Fig.3). Similarly, a quadrilateral in a snark can be replaced by two edges as shown in Fig.4 to produce a smaller snark. Therefore a non-trivial snark should have girth 5 or more.

It has also become standard to exclude snarks that contain an edge-cut consisting of two or three independent edges. It is easy to see that using such a cut a snark can be split into two smaller cubic graphs at least one of which is a snark (see Fig.5 and Fig.6). In other words, a snark with an independent (or equivalently, cycle-separating) 2-cut or 3-cut can be either 'reduced' to a smaller snark or 'decomposed' to two smaller snarks. In this manner, snarks with small cycle-separating edge-cuts also become 'trivial' modifications of those with fewer vertices. Therefore, a 'non-trivial' snark should be cyclically 4-edge-connected.

However, it has recently been shown that similar results hold for cut-sets of size 4 and 5 (see [2, 10, 21]). The following two theorems are the main results of Cameron, Chetwynd and Watkins [2, Theorem 1 and Theorem 2].

Theorem 1. *Let G be a snark with a cut-set of four edges whose removal leaves components G_1 and G_2. Then either*

(a) *one of G_1 and G_2 is not 3-edge-colourable; or*

(b) *both G_1 and G_2 can be "extended" to snarks by adding two vertices to one of them and edges to both in such a way that G is their dot-product.*

Recall that the dot-product of two snarks G and H is defined as follows. Take two non-adjacent edges $e = a_1a_2$ and $f = b_1b_2$ in G, and remove e and f from G. Further, take an edge xy in H whose neighbours are vertices a'_1 and a'_2 adjacent to x and vertices b'_1 and b'_2 adjacent to y, and remove x and y from H. To obtain the dot-product of G and H, join a_i to a'_i and b_i to b'_i, $i = 1, 2$. The resulting graph $G.H$ is a snark ([13, 2]).

Theorem 2. *Let G be a snark with a cut-set of five edges whose removal leaves components G_1 and G_2. Let G not be the Petersen graph. Then either*

(a) *one of G_1 or G_2 is not 3-edge-colourable; or*

(b) *both G_1 and G_2 can be "extended" to cubic graphs that are not 3-edge-colourable, by adding at most five vertices; at least one of these extended graphs is smaller than G.*

At the first glance one could deduce from the above decomposition results that apart from the Petersen graph snarks with cycle-separating 4-cuts and 5-cuts could also be considered trivial (cf. [2, 24, 25]). However, if we apply these theorems to the Isaacs flower snark I_5 (see Fig.10) we see that for each decomposition in the sense indicated above I_5 itself appears as one of the factors. Does not this mean that along with the Petersen graph, I_5 should also be included among non-trivial snarks?