# PERTURBATION BOUNDS FOR THE OPERATOR ABSOLUTE VALUE

Rajendra Bhatia

Indian Statistical Institute,
New Delhi 110 016, India
and
Sonderforschungsbereich 343,
Universität Bielefeld, Bielefeld, Germany

Abstract. It is shown how to estimate the norms of the derivatives (of all orders) of the map that takes an invertible operator to the positive part in its polar decomposition. Using this, perturbation bounds of any order can be obtained for this map.

**Perturbation Bounds for the Operator Absolute Value.**

Let $A$ be a bounded linear operator on a Hilbert space $\mathcal{H}$ . Let $|A| = (A^*A)^{1/2}$ be the **positive part** or the **absolute value** of $A$ .

In this note we show how to derive inequalities of the type

$$(1) \qquad || \, |A| - |B| \, || \leq \sum_{n=1}^{N} a_n \; ||A-B||^n + O(||A-B||^{N+1}),$$

where $A$ is an invertible operator, $B$ an operator close to it, $N$ is any positive integer and $a_n$ are coefficients which can be explicitly determined.

For $N = 2$ this was done in [1] and the approach used in that paper is developed further here.

Let $\mathcal{B}(\mathcal{H})$ denote the space of all linear operators on $\mathcal{H}$ and let $\mathcal{B}_{\text{inv}}(\mathcal{H})$, $\mathcal{B}_s(\mathcal{H})$ and $B_+(\mathcal{H})$ denote its subsets consisting of the invertible, the self–adjoint and the positive operators, respectively. Let $\varphi : \mathcal{B}_{\text{inv}}(\mathcal{H}) \to \mathcal{B}_+(\mathcal{H})$ be the map $\varphi(A) = |A|$ . Let $D^n\varphi(A)$ denote the nth order (Frechét) derivative of $\varphi$ . Let

$$a_n = ||D^n\varphi(A)||.$$

Then by Taylor's Theorem [3, Ch. 8] we have the inequality (1). So, our problem is reduced to estimating $||D^n\varphi(A)||$ for all $n$ .

Now $\varphi = f \circ g$, where $g(A) = A^*A$ and $f(A) = A^{1/2}$ is the positive square root of a positive operator A. We will study $f$ and $g$ separately and then combine the information obtained.

More generally, let $f$ be any function mapping $(0,\infty)$ into itself. This induces a map on $\mathcal{B}_+(\mathcal{H})$ which, for convenience, is again denoted by $f$ . Let $f^{(n)}$ be the (ordinary) n–th derivative of $f$ when it is viewed as a map on $(0,\infty)$ , and let $D^nf(A)$ be its n–th order Frechét derivative at $A$ when $f$ is viewed as a map on $\mathcal{B}_+(\mathcal{H})$ . If

$$(3) \qquad ||D^nf(A)|| = ||f^{(n)}(A)|| \quad \text{for all } A \in \mathcal{B}_+(\mathcal{H}),$$

we will say that $f$ is in the class $\mathcal{D}_n$ . The following Proposition is crucial for our analysis.

**Proposition 1.** Let $f$ be an operator monotone function. Then $f \in \bigcap_{n=1}^{\infty} \mathcal{D}_n$ .

*Proof.* In [1, Equations (10), (13)] we showed that an operator monotone function satisfies (3) for $n = 1, 2$ . The same argument will show that this is the case for all $n$ . First note that if $h(A) = A^{-1}$ then

$$(4) \qquad \begin{aligned} &D^nh(A)\,(B_1, B_2, \ldots, B_n) \\ &= (-1)^n \sum_{\sigma} A^{-1}\, B_{\sigma(1)} A^{-1}\, B_{\sigma(2)}\, A^{-1} \ldots A^{-1}\, B_{\sigma(n)}\, A^{-1}, \end{aligned}$$