Iterated Monoidal Categories

Z. Fiedorowicz, R. Schwänzl, and R. Vogt

Introduction

For many years it has been known that there is a strong connection between coherence theory of categories and coherence problems in homotopy theory. Early work of Stasheff [15] and MacLane [8] showed that monoidal categories are analogous in a precise way to 1-fold loop spaces. Later a similar connection was noted between symmetric monoidal categories and infinite loop spaces. This connection was exploited with great success in algebraic K-theory. For instance the group completion of the nerve of a symmetric monoidal category is an infinite loop space, and the homotopy groups of this infinite loop space are the Quillen K-groups of that category, which provide algebraic information about the original category. Conversely this fact has also been used to construct new examples of infinite loop spaces and infinite loop maps of great interest to topologists.

It is striking to note that there appears to be a very similar connection between braided tensor categories and 2-fold loop spaces. It is shown in [4] that the group completion of the nerve of a braided tensor category is a 2-fold loop space. However the proof of this fact given there, while short and simple, is in some ways mysterious and unsatisfactory. For the proof notes that it is already known that the group completion of the nerve of a free braided tensor category is a free 2-fold loop space, and the general result follows from this. The result also raises new questions such as: what algebraic structure on a category corresponds to an n-fold loop structure for $2 \leq n < \infty$?

This paper gives another proof of the aforementioned result and sets up a framework for dealing with questions like this. The key idea is a generalization of the notion of braided tensor category, whose algebraic structure is more transparently analogous to the structure of a 2-fold loop space. This notion we call a *2-fold monoidal category.*

For a 2-fold loop space can be thought of as a loop space in the category of loop spaces. Noting the analogy between monoidal categories and 1-fold loop spaces, the analogous categorical concept is a category which is monoidal in the category of monoidal categories. This gives the notion of a 2-fold monoidal category. It is easy to see that a braided tensor category is a special case of this notion. Moreover it is easy to adapt the Segal-Thomason proof that the group completion of the nerve of a symmetric monoidal category is an infinite loop space, to prove the same connection between 2-fold monoidal categories and 2-fold loop spaces. These constructions iterate in an obvious manner to give notions of n-fold monoidal categories corresponding to n-fold loop spaces.

This paper is a preliminary draft which raises more questions than it answers and is meant to call attention to problems in this area. We will only discuss the 2-fold case in any detail. In the last section we will briefly sketch the generalization to n-fold monoidal categories for $n \geq 3$.

Basic Definitions

Definition 1: A *(strict) monoidal category* is a category $\mathcal{C}$ together with a functor $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object E such that

1. $\Box$ is strictly associative.
2. E is a strict 2-sided unit for $\Box$.

A monoidal functor $(F, \eta) : \mathcal{C} \to \mathcal{D}$ between monoidal categories consists of a functor F such that $F(E) = E$ together with a natural transformation

$$\eta_{A,B} : F(A)\Box F(B) \to F(A\Box B),$$

which satisfies the following conditions

1. Internal Associativity: The following diagram commutes

$$\begin{tikzcd}
F(A)\Box F(B)\Box F(C) \arrow[r, "\eta_{A,B}\Box id_{F(C)}"] \arrow[d, "id_{F(A)}\Box\eta_{B,C}"] & F(A\Box B)\Box F(C) \arrow[d, "\eta_{A\Box B,C}"] \\
F(A)\Box F(B\Box C) \arrow[r, "\eta_{A,B\Box C}"] & F(A\Box B\Box C)
\end{tikzcd}$$

2. Internal Unit Conditions: $\eta_{A,E} = \eta_{E,A} = id_{F(A)}$.

Given two monoidal functors $(F, \eta) : \mathcal{C} \to \mathcal{D}$ and $(G, \zeta) : \mathcal{D} \to \mathcal{E}$, we define their composite to be the monoidal functor $(GF, \xi) : \mathcal{C} \to \mathcal{E}$, where ξ denotes the composite

$$GF(A)\Box GF(B) \xrightarrow{\zeta_{F(A),F(B)}} G\big(F(A)\Box F(B)\big) \xrightarrow{G(\eta_{A,B})} GF(A\Box B).$$

(It is an exercise to check that ξ satisfies the associativity condition above.) We denote by $\mathbf{M}onCat$ the category of monoidal categories and monoidal functors. Note that this the usual product in **Cat** defines a product in **MonCat**.

Definition 2: A 2*-fold monoidal category* is a monoid in $\mathbf{M}onCat$. This means that we are given a monoidal category $(\mathcal{C}, \Box, E)$ and a monoidal functor $(\boxdot, \eta) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which satisfies

1. External Associativity: the following diagram commutes in $\mathbf{M}onCat$

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "(\boxdot,\eta)\times id_{\mathcal{C}}"] \arrow[d, "id_{\mathcal{C}}\times(\boxdot,\eta)"] & \mathcal{C}\times\mathcal{C} \arrow[d, "(\boxdot,\eta)"] \\
\mathcal{C}\times\mathcal{C} \arrow[r, "(\boxdot,\eta)"] & \mathcal{C}
\end{tikzcd}$$

2. External Unit Conditions: the following diagram commutes in $\mathbf{M}onCat$

$$\begin{tikzcd}
\mathcal{C}\times E \arrow[r, "\subseteq"] \arrow[d, "\cong"] & \mathcal{C}\times\mathcal{C} \arrow[d, "(\boxdot,\eta)"] & E\times\mathcal{C} \arrow[l, "\supseteq"'] \arrow[d, "\cong"] \\
\mathcal{C} \arrow[r, "="] & \mathcal{C} & \mathcal{C} \arrow[l, "="']
\end{tikzcd}$$

Explicitly this means that we are given a second associative binary operation $\boxdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, for which E is also a two-sided unit. Moreover we are given a natural transformation

$$\eta_{A,B,C,D} : (A \boxdot B)\Box(C \boxdot D) \to (A\Box C) \boxdot (B\Box D).$$