On bounded treewidth duality of graphs

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April 7, 1994

Abstract

We prove that for any integers m, k, there is an integer n_0 such that if G is a graph of girth $\geq n_0$ then any partial k-tree homomorphic to G is also homomorphic to C_{2m+1}. As a corollary, every non-bipartite graph does not have bounded treewidth duality.

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1 Introduction

Any coloring (or scheduling) problem may be expressed as an appropriate graph-homomorphism problem. Recall that given two graphs $G = (V, E)$ and $G' = (V', E')$, a *homomorphism* $G \to G'$ is any map $f : V \to V'$ satisfying

$$(x, y) \in E \Longrightarrow (f(x), f(y)) \in E'.$$

Thus the question whether there exists a homomorphism of G to K_k (the complete graph with k vertices) is equivalent to the question whether $\chi(G) \leq k$. By fixing a target graph H (instead of the complete graph K_k) this leads to the following H-coloring problem:

Instance: Graph G

Question: Does there exist a homomorphism $G \to H$?

The complexity of H-coloring problem was completely solved for undirected graphs by Hell and Nesetril [12], while for directed graphs the problem seemingly to be presently intractable: only partial results are known, [2, 3, 4, 5, 8, 9, 10, 13, 14, 15, 22, 27]. This inherent difficulty of H-coloring problem for directed graphs was recently related to the constrained satisfaction problem and Datalog descriptions by Feder and Vardi, [7].

The following notion has been suggested in [13] as a possible approach to polynomial H-coloring problems:

Definition 1 *An H-coloring problem is said to have* bounded treewidth duality *if there exists a positive integer $k = k(H)$ such that the following holds:*

G is homomorphic to H if and only if every graph F homomorphic to G with treewidth $\leq k$ is also homomorphic to H.

In other words: G fails to be H-colorable if and only if there exists a graph F with treewidth $\leq k$ such that $F \to G$ and F fails to be H-colorable.

Denoting the non-existing of homomorphisms by $\not\to$ and treewidth of G by $tw(G)$, we can schematically express the duality as

$$G \not\to H \quad \Longleftrightarrow \quad \exists F, F \to G, F \not\to H, \mathrm{tw}(G) \leq k.$$