OPTIMAL RATES OF CONVERGENCE IN THE CLT FOR QUADRATIC FORMS

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Abstract. We prove optimal convergence rates in the Central Limit Theorem for sums in $\mathbf{R}^k$. Assuming a fourth moment we obtain a Berry-Esseen type bound of order $O(N^{-1})$ for the probability to hit a ball provided that $k \geq 5$. The proof still requires a technical assumption related to the independence of coordinates of sums.

1. Introduction and results

Let $\mathbf{R}^k$ denote the real k-dimensional Euclidean space with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$, $1 \leq k < \infty$. For $k = \infty$ regard $\mathbf{R}^k$ as a real separable Hilbert space.

Let $X, X_1, X_2, \dots$ be a sequence of independent and identically distributed (i.i.d.) random variables taking values in $\mathbf{R}^k$. Write

$$S_N = (X_1 + \cdots + X_N)/\sqrt{N}.$$

Assume that $\mathbf{E}X = 0$ and that $\mathbf{E}|X|^2 < \infty$. Then X is pregaussian, i.e. there exists a centered Gaussian random variable G such that the covariances of X and G are equal.

Let us consider a quadratic form,

$$Q(x) = (Qx, x), \qquad \text{for} \quad x \in \mathbf{R}^k,$$

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where $Q : \mathbf{R}^k \to \mathbf{R}^k$ denotes a symmetric linear bounded operator.

Assuming fourth moment $\mathbf{E}|X|^4 < \infty$ we shall prove the following estimate of the convergence rate in the Central Limit Theorem,

$$\sup_r \Big| \mathbf{P}\{Q(S_N) < r\} - \mathbf{P}\{Q(G) < r\} \Big| = O(N^{-1}). \tag{1.1}$$

It is important to emphasize that we do *not* require any smoothness conditions like Cramer's (C) condition for the characteristic function of X or its coordinates. Related lower bounds in the lattice point problem (see, for instance, Fricker (1982)) suggest that (1.1) may not be valid in dimensions smaller than five. We shall prove that (1.1) holds for $k \geq 5$. For dimension $k = 4$ we get $O(N^{-1} \ln^\delta N)$ with some $\delta > 0$. The dimensions $k = 2, 3$ are not studied here since they require different methods. Unfortunately, in order to prove (1.1) we still need certain assumptions on the independence of coordinates of X. For example, the following conditions are sufficient: the first five coordinates of X have positive variances, are mutually independent and do not depend on other coordinates; the operator Q is diagonal and its first five eigenvalues are non-zero.

Let us formulate the result more precisely. Without loss of generality we shall assume that X is not concentrated in a proper subspace of $\mathbf{R}^k$, i.e. that $\ker B = \{0\}$, where the covariance operator B^2 of X is defined by

$$(B^2 x, y) = \mathbf{E}(X, x)(X, y) = \mathbf{E}(G, x)(G, y), \quad \text{for all} \quad x, y \in \mathbf{R}^k.$$

In the finite dimensional case $\ker B = \{0\}$ implies that B has rank k. In the case $k = \infty$ the operator B^2 is nuclear, and $\ker B = \{0\}$ implies that all eigenvalues of B^2 are positive.

Without loss of generality we may assume that $\ker Q = \{0\}$, which means that $\mathbf{R}^k = \overline{\mathrm{Im} Q}$ (the closure of the image of Q) since Q is symmetric. Otherwise we may replace X by its projection onto $\overline{\mathrm{Im} Q}$ without changing the distribution of (QX, X), and without violating other conditions on X.

A consequence of these assumptions is that $\ker BQB = \{0\}$.

The independence condition. We shall assume that the random variable X is a sum of two independent random variables,

$$X = U + V, \qquad \text{where} \quad U \quad \text{and} \quad V \quad \text{are independent.} \tag{1.2}$$

Furthermore, we require that U, $\mathbf{E}U = 0$, assumes values in an invariant subspace of Q, say $\mathbf{R}_X$, with the dimension at least 5, that Q is diagonal in some orthonormal basis of $\mathbf{R}_X$, and that the coordinates of U in this basis are independent and have