The approximation theorem and the K-theory of generalized free products

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1 Introduction

One of the main results of [8] is a decomposition of the K-theory of a ring which is a generalized free product in terms of the K-theories of the constituent rings, and one other object, which is generally called the Nil-term. We recall this in the fibration square described below in Theorem 4. Underlying the derivation of the square given in [8] is the construction of a suitable fibration. Besides a combinatorial analysis of a generalized free product, the construction depends on an *ad hoc* development of some parts of abstract algebraic K-theory, later systematized and generalized in [9]. It is the purpose of this paper to show how to derive the main theorems of [8] as applications of results and methods of [9]. Using different methods, Pierre Vogel [7] has also reconsidered results of [8].

We first develop language so that essentially the same fibration used in [8] may be derived from a general fibration theorem (Theorem 1.6.4 of [9, page 354]) developed as part of the overall approach to abstract algebraic K-theory described in [9]. This is Proposition 2.1 below. Our main contribution is the description of how the approximation theorem (Theorem 1.6.7 of [9, page 354]) may then be used to interpret terms in the fibration. These results are Theorems 2 and 3 stated at the end of the section. We are able to replace the technical manouvering required in the original proofs with arguments that follow a standard pattern and are more conceptual. The paper also provides an introduction to a few of the ideas we will use in [5], where we generalize the situation to the case of simplicial rings. One of the goals of [5] is to set up a framework which will also allow us to handle decomposition problems in the K-theory of A_∞-rings (informally, rings-up-to-homotopy), including partial analysis of the Nil terms. To these ends, several parts of the program in the present paper have to be treated differently in [5], with the result that the specialization of those proofs to the case of ordinary rings does not immediately make the underlying principles clear.

We will concern ourselves only with the case of a generalized free product of rings, but it will be clear that the methods also reprove the results for Laurent extensions and for polynomial extensions which are found in sections 12 and 13 of [8], respectively.

We start with a diagram

$$A \xleftarrow{\alpha} C \xrightarrow{\beta} B$$

of discrete rings satisfying the freeness and purity conditions of [8]. That is, we require for purity that $\alpha : C \longrightarrow A$ be injective and that there be a splitting $A = \alpha(C) \oplus A'$ of C-bimodules. Though the splitting is not part of the data and only its existence is required, it is convenient to refer to a fixed complement A' of $\alpha(C)$ in A. The freeness condition we will impose is that A' shall be free as a left C-module. We impose the same conditions on $\beta : C \longrightarrow B$.

Let

$$R = A *_C B$$

be the free product as displayed in [8]. As a C-bimodule, then

$$R = C \oplus A' \oplus B' \oplus A' \otimes_C B' \oplus B' \otimes_C A' \oplus A' \otimes_C B' \otimes_C A' \oplus B' \otimes_C A' \otimes_C B' \oplus \cdots$$

and the problem is to describe the K-theory of R in terms of the K-theories of A, B, and C.

In this paper K-theory is defined in terms of the $S_\bullet$ construction of [9], which from a category with cofibrations and weak equivalences produces its K-theory space. We consider certain categories of simplicial modules over the rings A, B, C, and R. These categories, whose definitions we recall in Section 2, will be denoted by $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, $\mathcal{M}_f(C)$, and $\mathcal{M}_f(R)$, respectively. Each of these categories then supports a notion of cofibration, which we also recall below, and a weak equivalence in each case will be a homotopy equivalence, or, briefly, an h-map. Then the K-theory of $\mathcal{M}_f(A)$ with respect to these notions of cofibration and weak equivalence (generically indicated by the presence of an h somewhere in the symbol) is defined in [9] as

$$K(\mathcal{M}_f(A); h) = \Omega|hS_\bullet\mathcal{M}_f(A)|.$$

According to Theorem 2.3.2 in [9, page 394] the relation with Quillen's plus construction definition of K-theory is

$$\Omega|hS_\bullet\mathcal{M}_f(A)| \simeq K_0'(A) \times BGL(A)^+,$$