SOME GLUEING FORMULAS FOR SPIN POLYNOMIALS AND A PROOF OF THE VAN DE VEN CONJECTURE.*

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Abstract. We deduce some formulas for the spin invariants of the connected sum of an arbitrary 4-manifold $X, b_2^+(X) > 0$ with $\overline{CP^2}$ in terms of the spin invariants of X and apply this to prove the Van de Ven conjecture.

§0. Introduction

In this article we present some formulas for the spin-polynomial of the connected sum of an arbitrary oriented smooth simply connected four-manifold X with $b_+^2(X) > 0$ with the orientation reversed complex projective plane. These formulas are similar to ones for Donaldson's polynomials presented by Donaldson ([D]) and Morgan-Mrowka ([MM]) in the sense that they do not involve evaluating ν-classes for the polynomial of X. One can show that these formulas also hold in the case of connected sums with arbitrary negative definite manifolds. However the case of $\overline{CP^2}$ is easier from the technical point of view.

As an application one has a proof of the Van de Ven conjecture [VdV]:

Conjecture. *The Kodaira dimension of a simply connected algebraic surface is an invariant of its smooth structure.*

This has been proved to be true for all cases except possibly nonminimal surfaces of general type (cf. [FM], [K], [OVdV], [P], [PT], [Q1], [Q2]). Therefore it suffices to prove that

No surface of general type can be diffeomorphic to a rational one.

The proof is a reduction to the case of minimal surfaces of general type, which already was treated in [PT]. This reduction uses the glueing formulas and results of [FM] describing diffeomorphisms of rational surfaces . In [FQ1] a different approach to the Van de Ven conjecture is announced. The authors use a glueing formula for Donaldson polynomials (and predict our approach) and also explicitly formulate the result on diffeomorphism of rational surfaces which we use.

§1. Connected sum formulas.

We shall start with some notations.

Let X be a Riemannian, simply-connected 4-manifold with a metric g and $E \to X$ some complex hermitian vector bundle of rank 2 with chern classes $c_1(E), c_2(E)$.

*This is the translation of an article submitted to Izvestija of Russian Academy of Sciences, Ser.Math. While this translation was in progress there appeared a paper [FQ2] with the proofs of the results announced in [FQ1].

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

This vector bundle as well as the corresponding principal $U(2)$ -bundle P is uniquely determined by its chern classes. Denote also by P_{ad} the $PU(2)$-principal bundle $P_{ad} = P/S^1$, and by adE the corresponding vector bundle.

We shall consider the space $\mathcal{A}_E$ of $U(2)$- connections on the bundle E together with the action of the gauge group $\mathcal{G}_{U(2)}$ and the space of orbits of this action $\mathcal{B}_E$. The space of $SO(3)$ -connections for the bundle adE, the gauge group, and the orbit space, we denote by $\mathcal{A}_{adE}, \mathcal{G}_{SO(3)}, \mathcal{B}_{adE}$ resp. We write $\mathcal{A}^*_{adE} \subset \mathcal{A}_{adE}$ and $\mathcal{B}^*_{adE} \subset \mathcal{B}_{adE}$ for the set of irreducible connections on adE and the orbits of irreducible connections.

We shall also need the space of orbits of framed connections $\widetilde{\mathcal{B}_{adE}}$, that is the orbit space of the space of connections by the subgroup $\mathcal{G}^0_G$ of the gauge group acting trivially on the fibre over some fixed point p.

Let

$$\mathcal{A}_\lambda = \{A \in \mathcal{A}_E \mid \frac{1}{2} trF_A = \lambda\}$$

for some closed differential 2-form λ on X. There is an isomorphism:

$$\mathcal{A}_\lambda / \mathcal{G}_{U(2)} = \mathcal{A}_{adE} / \mathcal{G}_{(SO(3)=PU(2))}. \tag{1.1}$$

Let C be some $Spin^{\mathbb{C}}$-structure on the manifold X. This means that one fixes an integral lift C of the second Stiefel-Whitney class $w_2(X)$ of X, and this lift defines a pair of complex hermitian rank 2 vector bundles W^+, W^- subject to the following conditions:

$$T^*_{\mathbb{C}} X = Hom_{\mathbb{C}}(W^+, W^-), \; c_1(\Lambda^2 W^{\pm}) = C.$$

A choice of a connection ∇ on the line bundle $\Lambda^2 W^+$ gives rise to a Dirac operator on X:

$$\mathcal{D}^{C,\nabla} : \Gamma(W^+) \to \Gamma(W^-).$$

Coupling it with connections $a \in \mathcal{A}_E$ then gives a family of Fredholm operators over $\mathcal{A}_E$:

$$\mathcal{D}^{C,\nabla}_a : \Gamma(W^+ \otimes E) \to \Gamma(W^- \otimes E),$$

which is equivariant with respect to the gauge group action. Using the isomorphism (1.1) we consider it as a family over $\mathcal{B}_{adE}$.

We shall always suppose that

$$ind(\mathcal{D}^{C,\nabla}_a) = dimker(\mathcal{D}^{C,\nabla}_a) - dimcoker(\mathcal{D}^{C,\nabla}_a) \leq 0.$$

Let $\mathbb{S}(\Gamma(W^+ \otimes E))$ denote the unit sphere in the Hilbert space $\Gamma(W^+ \otimes E)$, and $\mathbb{P}(\Gamma(W^+ \otimes E))$ the projectivisation of this space. There is a tautological bundle $\mathcal{O}_{\mathbb{P}(\Gamma(W^+ \otimes E))}(1)$ over this projective space, and the coupled Dirac operator can be interpreted as a section of the bundle

$$\Gamma(W^- \otimes E) \otimes \mathcal{O}_{\mathbb{P}(\Gamma(W^+ \otimes E))}(1).$$

The space

$$\mathcal{P} = \mathcal{A}_\lambda \times_{\mathcal{G}_{U(2)}} \mathbb{S}(\Gamma(W^+ \otimes E)) \tag{1.2}$$