Normality for Elementary Subgroup Functors

By Anthony BAK and Nikolai VAVILOV[1]

*Department of Mathematics, University of Bielefeld,
Postfach 10 01 31, 33501 Bielefeld, Germany*

and

*Department of Mathematics and Mechanics, University of Sanct–Petersburg,
Petrodvorets, 198904, Russia*

Abstract We define a notion of group functor G on categories of graded modules, which unifies previous concepts of a group functor G possessing a notion of elementary subfunctor E. We show under a general condition which is easily checked in practice that the elementary subgroup $E(M)$ of $G(M)$ is normal for all quasi-weak Noetherian objects M in the source category of G. This result includes all previous ones on Chevalley and classical groups G of rank $\geqq 2$ over a commutative or module finite ring M (since such rings are quasi-weak Noetherian) and settles positively unanswered cases of normality for these group functors.

1. Introduction.

Beginning with H. Bass' pioneering paper [8], much attention has been given to determining whether or not the elementary subgroup $E(M)$ of a Chevalley or classical group $G(M)$ over a module finite ring M is normal in $G(M)$. Bass' result [8] was valid for $G = GL_n$ (general linear group of rank n) under the condition that n was large with respect to the dimension of the ring M. It was immediately asked whether or not similar looking subgroups of other classical groups were also normal. This question was settled by H. Bass [9] in the symplectic case and in general by A. Bak [3], [4], again under the assumption that the rank of the functor was large with respect to the dimension of the ring. The approach taken in [3], [4] introduced a generalization GQ_{2n} (referred to by the notation U_{2n} in [3], [4], [9] and by GQ_{2n} in [5]) of

[1] The second-named author gratefully acknowledges the support of the *Alexander von Humboldt-Stiftung* and the hospitality of the *Universität Bielefeld* during the preparation of the present article.

the notion of classical group, which depended on the concept of a form parameter and allowed treating in a uniform way all the classical and classical-like groups other than those in the GL_n-family. The groups corresponding to the minimum form parameter are treated also in L. Vaserstein [23]. After a gap of roughly ten years, A. Suslin showed in [17], [22] that the elementary subgroup $E_n(M)$ of $GL_n(M)$ is normal whenever $n \geq 3$ and M is commutative or module finite. Following his results, several papers Kopeiko [13], Suslin-Kopeiko [18], Taddei [20], [21], Golubchik-Mikhalev [11] and Vavilov [25] were devoted to analogous questions for Chevalley groups and general quadratic groups GQ_{2n}. A detailed survey of developments to-date is contained in [12], [26] and [27]. Many subsidiary references can be found there. The publications above leave open the case of twisted Chevalley groups and the general case (in classical language, nonsplit case) of the general quadratic group $GQ_{2n}(M, \Lambda)$ ($n \geq 2$) for an arbitrary module finite ring M with involution and form parameter Λ. The twisted Chevalley groups of type $\neq A_{2\ell}$ are settled in (3.12) and the general quadratic group is settled in (3.10), both as consequences of our Main Theorem (2.9). The result (3.10) is included in those announced in [6, §1].

The approach we take to prove our results is inspired by the technique used in [6], particularly in [6, §4], for establishing results concerning GL_n. This technique hinges for the most part only on a simple minded, flexible, and general concept of group functor, which is divorced from the confines of linear algebra and at the same time contains as special cases the group functors in the previous paragraph. Specifically, we consider group functors G on a category $\mathbf{\Lambda}$ of graded modules M (for example, a category of rings or form rings) such that $G(M)$ contains the homogeneous components of M in a functorial way. The **elementary subgroup** $E(M)$ of $G(M)$ is defined as the subgroup of $G(M)$ generated by the embedded homogeneous components of M. This construction obviously defines a subfunctor $E \rightarrowtail G$, which we christen the **elementary subfunctor** of G. In the case of Chevalley or general quadratic groups G, it is clear that the notion of an elementary matrix defines several embeddings into G of the ring and form parameter over which G is defined and that the elementary subgroup of G is indeed the subgroup generated by these embedded entities. On the other hand, any Chevalley or general quadratic group is itself embedded set theoretically, but still functorially, into a direct sum of copies of the ring over which it is defined. So, we assume in the general setting that our groups $G(M)$ embed set theoretically and functorically into a direct sum $\oplus M$ of copies of M. This extra structure is enough to prove in just a few words our main result (2.9) asserting the normality of $E(M)$ in $G(M)$ whenever M is quasi-weak Noetherian (2.6). It should be noted that when the domain