THE CONJECTURE OF LANGLANDS AND RAPOPORT FOR CERTAIN SHIMURA VARIETIES OF NON-RATIONAL WEIGHT

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Abstract. The conjecture of Langlands and Rapoport provides a precise description of the points on a Shimura variety modulo a prime of good reduction. We refine the conjecture so that it also describes the points on the reduction of each connected component of the Shimura variety, and we prove (under some hypotheses) that the validity of the refined conjecture for a Shimura variety depends only on the associated connected variety. Using that the conjecture is known for Shimura varieties of PEL-type, we deduce the conjecture for other Shimura varieties, including some whose weight is not defined over $\mathbb{Q}$.

Introduction

0.1. The starting point of Langlands's program, to express the zeta function of a Shimura variety in terms of automorphic L-series, is the conjecture of Langlands and Rapoport, see 5.1e in [L.-R.'87].

A Shimura variety is defined by a reductive group G acting on a Hermitian symmetric domain X, and is known to have a canonical model $\mathrm{Sh}(G, X)$ over a certain number field E, called the reflex field. It is conjectured that the quotient $\mathrm{Sh}_p(G, X)$ of $\mathrm{Sh}(G, X)$ by a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$ has good reduction at any prime v of E lying over p. The set of points $\mathrm{Sh}_p(\mathbb{F})$ of the reduced variety with coordinates in the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ has an action of the Frobenius automorphism and of the group $G(\mathbb{A}_f^p)$, and the conjecture of Langlands and Rapoport gives an explicit description of $\mathrm{Sh}_p(\mathbb{F})$, together with these two actions, in terms of the data (G, X, K_p).

The form of the conjecture is suggested by Deligne's "dream" that all Shimura varieties with rational weight should be moduli varieties for motives with additional structure. Thus $\mathrm{Sh}_p(\mathbb{F})$ should parameterize motives over $\mathbb{F}$ with additional structure. The motives over $\mathbb{F}$ form a Tannakian category, and Langlands and Rapoport define a groupoid, which they call the pseudomotivic groupoid $\mathfrak{P}$, and which they conjecture to be the groupoid attached to the category of motives over $\mathbb{F}$. For Shimura varieties with rational weight, their conjecture is expressed in terms of $\mathfrak{P}$.

The Shimura varieties whose weight is not rational can not be moduli varieties for motives, and in order to state a conjecture for these Shimura varieties, Langlands and Rapoport replace $\mathfrak{P}$ with a new groupoid, which they call the quasimotivic groupoid $\mathfrak{Q}$. Unfortunately, the groupoid they construct does not have the prop-

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erties they claim[1]. In §2.3 of this paper, we define a new quasimotivic groupoid $\mathfrak{Q}$, different from that of Langlands and Rapoport, which allows us to state the conjecture for all Shimura varieties.

One deficiency of the conjecture is that it does not provide a description of the points on the various connected components of Sh_p. We remedy this in §3.2 by stating a conjecture that describes the map

$$\mathrm{Sh}_p(\mathbb{F}) \to (\pi_0 \mathrm{Sh}_p)(\mathbb{F})$$

together with the actions of the various operators. We refer to this as the *refined Langlands-Rapoport conjecture.* In fact, for the statement to be useful, it is necessary to replace the group $G(\mathbb{A}_f^p)$ by a larger group $\mathcal{G}_p$. In [Pf.'93], we showed that Milne's proof of the original conjecture for Siegel modular varieties extends to give a proof of the refined conjecture.

With the refined conjecture, it is sometimes possible to pass from one Shimura variety to a second when their associated connected Shimura varieties are isomorphic. In fact, the main result of this paper is the following:

Theorem. *Let $\mathrm{Sh}_p(G, X)$ and $\mathrm{Sh}_p(G_1, X_1)$ be Shimura varieties defined by triples such that*

$$(G^{\mathrm{der}}, X^+, K \cap G^{\mathrm{der}}(\mathbb{Q}_p)) \approx (G_1^{\mathrm{der}}, X_1^+, K_1 \cap G_1^{\mathrm{der}}(\mathbb{Q}_p)).$$

Assume that the centers of G and G_1 have trivial first Galois cohomology groups and that the derived groups G and G_1 are simply connected. If the refined conjecture is true for one of the Shimura varieties, then it is true for the other.

The following result of Milne shows that it is of little importance that, in the above theorem, we require the group G to have simply connected derived group. In order to obtain the conjecture of Langlands and Rapoport for all Shimura varieties it suffices to consider only those arising from groups with simply connected derived group, see 4.18 [Mi.'92a]. As an interesting application of the above theorem we get the fact that the refined conjecture of Langlands and Rapoport for Shimura varieties of PEL-type implies the same for Deligne's strange models.

We now explain the organization of the article. We first need to have a good understanding of canonical models of connected and non-connected Shimura varieties and their relationship. In §1.1 - §1.3 we develop that background material. We compute the set of connected components $\pi_0(\mathrm{Sh}_p)$ and we show that it is more canonical to replace $G(\mathbb{A}_f^p)$ by a slightly larger group $\mathcal{G}_p(G)$. In §2.1 we recall basic facts about groupoids. Let S denote the Serre group. In §2.2 we recall the construction of the pseudomotivic groupoid $\mathfrak{P}$ and we recall the morphism

$$\mathfrak{P} \longrightarrow \mathfrak{G}_S,$$

[1]Specifically, on p.144 of their article, they define an element γ and claim that $\lambda(\gamma)$ is a quasi-Weil number. However, they do not check the condition at ∞, and in fact, it does not satisfy that condition.