# MODULI FIELDS OF CM-MOTIVES APPLIED TO HILBERT'S 12-TH PROBLEM

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Abstract. We apply the theorem of Shimura and Taniyama to compute the largest abelian extension of a CM-field $E$ which can be obtained by evaluating $E$-rational automorphic functions at certain special points of Shimura varieties.

Abelian class field theory classifies the abelian extensions of a number field $k$, but does not explain how to generate the fields. In his Jugendtraum, Kronecker suggested that the abelian extensions of $\mathbb{Q}$ can be generated by special values of the exponential function, and that the abelian extensions of an imaginary quadratic number field can be generated by special values of elliptic modular functions. This idea of generating abelian extensions using special values of holomorphic functions was taken up by Hilbert in his twelfth problem, where he suggested "finding and discussing those functions that play the part for any algebraic number field corresponding to that of the exponential function for the field of rational numbers and of the elliptic modular functions for imaginary quadratic number fields."

In this paper we determine the extensions of an algebraic number field that can be obtained by adjoining the special values of automorphic functions, or, more precisely, by the special values of rational functions on Shimura varieties whose weight is defined over $\mathbb{Q}$. When the Shimura variety is a moduli variety for polarized abelian varieties, the fields that can be generated in this way are the fields of moduli of polarized abelian varieties with complex multiplication. For more general Shimura varieties, they are the fields of moduli for CM-motives with Hodge cycle structure.

Our main result (Theorem 2.1) can be stated as follows: Let $E$ be a CM-field, i.e., a totally imaginary quadratic extension $E$ of a totally real field $F$, and let $\mathcal{M}_E$ be the field generated over $E$ by the fields of moduli of all CM-motives with Hodge cycle structure whose reflex field is contained in $E$; then $\mathcal{M}_E$ is the subfield the maximal abelian extension $E^{\mathrm{ab}}$ of $E$ fixed by the image of $\mathrm{Gal}(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}})$ under the Verlagerung map. In fact, we show that in order generate $\mathcal{M}_E$ it suffices to take abelian varieties of CM-type rather than CM-motives. As a consequence we recover the results of Shimura [Sh.'62] and Ovseevich [Ov.'74] that $E^{\mathrm{ab}}$ can be obtained from $F^{\mathrm{ab}}\cdot\mathcal{M}_E$ by the adjunction of (infinitely many) square roots. The key tool in the proof of the theorem is an extension of the theorem of Shimura and Taniyama to CM-motives.

In section 1, we identify $\mathcal{M}_E$ as the subfield of $E^{\mathrm{ab}}$ fixed by the kernel of Deligne's reciprocity law homomorphism

$$r_E : \mathrm{Gal}(E^{\mathrm{ab}}/E) \to S^E(\mathbb{A}_f)/S^E(\mathbb{Q}),$$

and we prove that to generate $\mathcal{M}_E$ it suffices to take fields of moduli of abelian varieties of CM-type. In the next section, we prove that the kernel of $r_E$ is the image of the Verlagerung map

$$\mathrm{Ver}: \mathrm{Gal}\,(F^{\mathrm{ab}}/F\cdot\mathbb{Q}^{\mathrm{ab}}) \longrightarrow \mathrm{Gal}\,(E^{\mathrm{ab}}/E),$$

and hence that $\mathcal{M}_E = (E^{\mathrm{ab}})^{\mathrm{Im(Ver)}}$. Because the composite

$$\mathrm{Gal}\,(F^{\mathrm{ab}}/F) \xrightarrow{\mathrm{Ver}} \mathrm{Gal}\,(E^{\mathrm{ab}}/E) \xrightarrow{\mathrm{restrict}} \mathrm{Gal}\,(F^{\mathrm{ab}}/F),$$

is multiplication by 2, it follows that $\mathrm{Gal}\,(E^{\mathrm{ab}}/\mathcal{M}_E\cdot F^{\mathrm{ab}})$ is killed by 2.

Finally, in the third section, we explain how fields of moduli of CM-motives are related to special values of automorphic functions.

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## 0. Notations And Conventions

The algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is denoted by $\overline{\mathbb{Q}}$ and all number fields are subfields of $\overline{\mathbb{Q}}$ of finite degree over $\mathbb{Q}$. The action of complex conjugation on $\mathbb{C}$ and $\overline{\mathbb{Q}}$ is denoted by $\iota$. The ring of finite adeles of $\mathbb{Q}$ is denoted by $\mathbb{A}_f$. The Weil restriction of scalars of $\mathbb{G}_m$ from $L$ to $\mathbb{Q}$ is denoted by $T^L$. Let $G$ be a $\mathbb{Q}$-rational reductive group. We denote by $G(\mathbb{Q})^+$ the intersection of $G(\mathbb{Q})$ with the identity component of $G(\mathbb{R})$. The inverse image of $G^{\mathrm{ad}}(\mathbb{Q})^+$ in $G(\mathbb{Q})$ is denoted by $G(\mathbb{Q})_+$. The closure of a subgroup $H$ of $T(\mathbb{A}_f)$ in $T(\mathbb{A}_f)$ is denoted by $H^\frown$. All modules and vector spaces are assumed to be free of finite rank over their ring of coefficients. With Hodge structures and associated data we follow Deligne's convention, see [De.'79]. The reciprocity homomorphism of class field theory

$$\mathrm{rec}_L\colon \mathbb{A}_L^\times \longrightarrow \mathrm{Gal}\,(L^{\mathrm{ab}}/L),$$

where $\mathbb{A}_L$ denotes the ring of adeles of $L$, is normalized so that it maps a local uniformizing element to the corresponding geometric Frobenius. When restricted to the group of finite ideles, it induces an isomorphism

$$T^L(\mathbb{A}_f)/T^L(\mathbb{Q})^{+\frown} \longrightarrow \mathrm{Gal}\,(L^{\mathrm{ab}}/L).$$

## 1. Fields Of Moduli For CM-Motives Over $\overline{\mathbb{Q}}$

**1.1. The Serre group.** Let $L$ be a number field. The Serre group $S^L$ for $L$ and its cocharacter $\mu^L$ have been characterized in the following three ways in [DMOS III].

**Proposition 1.1.**

(1) *For every pair $(T,\mu)$ consisting of a $\mathbb{Q}$-rational torus $T$ with cocharacter $\mu$ defined over $L$ satisfying the Serre condition, namely,*

$$(\iota+1)(\sigma-1)\mu = 0 = (\sigma-1)(\iota+1)\mu \qquad \forall\sigma\in\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}),$$