Eisenstein series, Green functions, and knots

by

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Introduction

In these notes we study a certain type of functions on three dimensional hyperbolic space $\mathbb{H}^3$ automorphic with respect to a cocompact group Γ acting on $\mathbb{H}^3$ which, from their origin, show a close relation to the classical real analytic Eisenstein series. These functions are not of Poincaré series type. We understand their existence as some analytic counterpart of the geometric manipulation which produces, via Dehn surgery, compact hyperbolic manifolds from those with finite volume. A cusp of a lattice Γ_0 in $\mathrm{Iso}^+(\mathbb{H}^3)$ will, when geometrically deforming the hyperbolic structure of $\Gamma_0\backslash\mathbb{H}^3$ become a one dimensional closed geodesic in $\Gamma\backslash\mathbb{H}^3$, so a knot with possibly self intersection. Here, Γ is the cocompact lattice which has been produced by deformation of Γ_0. The functions in question, when regarded as functions on the manifold $\Gamma\backslash\mathbb{H}^3$, will be real analytic complex valued outside such a knot, and they will carry typical singularities along these knots: They will serve, inside the space of smooth functions on $\Gamma\backslash\mathbb{H}^3$, as Green functions in a non classical understanding. Traditionally, Green functions do have singularities concentrated in isolated points, and they, when in a typical way, put as kernel into an integral operator, achieve the evaluation of smooth functions at these singularities. One may look at them as linear operators on smooth function spaces which come from integration along (discrete) zero dimensional subsets of $\Gamma\backslash\mathbb{H}^3$. The functions studied here will do something similar: They may be looked at as functionals on smooth function spaces on $\Gamma\backslash\mathbb{H}^3$ which are created by integrating along geodesic knots in $\Gamma\backslash\mathbb{H}^3$.

This point of view explains why there is no satisfying analogy in dimension 2; Dehn surgery in dimension 2 simply amounts to application of Riemann's theorem on removability of singularities. Analogies in dimension 4 or higher may exist. They have not been studied, yet. We have, instead, been attracted by the idea that a new light might be shed upon knot theory, of course, upon knot theory in the three sphere as well as in general closed three dimensional manifolds. This light might back the philosophy that knot theory is not just algebra but largely geometry and analysis.

We freely make use of standard notations and formalisms in three dimensional (Lobačevskij) hyperbolic geometry: $\mathbb{H}^3 = \mathbb{C}\times\mathbb{R}_{>0}$ is three dimensional upper half space with boundary $\partial\mathbb{H}^3 = P^1\mathbb{C}$ the Riemann sphere. $PSL_2(\mathbb{C})$ is the group $\mathrm{Iso}^+(\mathbb{H}^3)$ of orientation preserving isometries of $\mathbb{H}^3$; the explicit