Cohen-Macaulay Modules over Cohen-Macaulay Algebras

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1 Introduction

This survay is devoted to three main subjects concerning tame and wild Cohen-Macaulay algebras (all of Krull dimension 1 and without nilpotent ideals):

1. Families of Cohen-Macaulay modules and "the number of parametres" in such families:
   (a) construction of "almost versal" families with projective bases;
   (b) definition of the "number of parametres";
   (c) semi-continuity of this number in families of algebras.
2. Tame and wild algebras:
   (a) definition via the "universality property";
   (b) definition via the number of parametres in families of modules;
   (c) tame/wild dichotomy and equivalence of definitions;
   (d) tame and wild algebras in families of algebras.
3. Criteria of tameness:

*Supported by Foundation for Fundamental Research of Ukraine and Deutsche Forschungsgemeinschaft.

(a) commutative case;

(b) case of 2×2 matrix rings.

For technical reasons we consider only the "geometrical situation", where all algebras are those over some base filed (usually supposed algebraically closed).

Most of these results have been obtained by G.-M. Greuel and the author. Here the proofs are mainly sketched (and sometimes omitted). For technical details we refer to [8]-[11].

2 Basic Definitions

2.1 Cohen-Macaulay algebras

Throughout this paper we use the following definitions and notations.

Definition 2.1.1 1. *Call a noetherian commutative ring* R Cohen-Macaulay ring *if all its localizations at prime ideals are Cohen-Macaulay local rings.*

2. *For a Cohen-Macaulay ring* R *denote* $\mathcal{CM}(\mathsf{R})$ *the category of* maximal Cohen-Macaulay R-modules M, *i.e. such (finitely generated) Cohen-Macaulay* R*-modules* M, *that* $\dim M_{\mathbf{m}} = \dim \mathsf{R}_{\mathbf{m}}$ *for each maximal ideal* $\mathbf{m} \subset \mathsf{R}$. *Later on we usually omit the word "maximal" in this description.*

3. *Call a ring* Λ Cohen-Macaulay algebra *if it is an algebra over some Cohen-Macaulay ring* R *and belong to* $\mathcal{CM}(\mathsf{R})$ *as* R*-module. To precise, if necessary, we call iit also* Cohen-Macaulay R-algebra.

4. *For a Cohen-Macaulay* R*-algebra* Λ *denote* $\mathcal{CM}(\Lambda)$ *the category of* (maximal) Cohen-Macaulay (left) Λ-modules, *i.e. those, which belong to* $\mathcal{CM}(\mathsf{R})$ *considered as* R*-modules.*

5. *We call a Cohen-Macaulay algebra (in particular, a Cohen-Macaulay ring)* reduced *if it contains no non-zero nilpotent ideals, and* analiticcally reduced *if, for each maximal ideal* $\mathbf{m} \subset \mathsf{R}$, *its* $\mathbf{m}$*-adique completion is also reduced.*