Properly discontinuous groups on certain matrix homogeneous spaces

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Abstract. We characterize discrete groups $\Gamma \subset GL(n, \mathbf{R})$ which act properly discontinuously on the homogeneous space $GL(m, \mathbf{R})\backslash GL(n, \mathbf{R})$.

§1. Introduction

A manifold M is called a complete locally homogeneous if $M = J\backslash H/\Gamma$. Here H is a finite dimensional Lie group, $J \subset H$ its closed Lie subgroup and $\Gamma \subset H$ is a discrete subgroup which acts freely and properly discontinuously on $J\backslash H$. See [**Gol**]. In the last thirty years there was a lot of activity in the case where J is noncompact (the nonclassical case). Consult our list of references.

The first basic problem in this area is to characterize all discrete subgroups $\Gamma \subset H$ which act freely and properly discontinously on $J\backslash H$. The most known example is Calabi-Markus phenomenon. That is, only finite groups can act freely and properly discontinuously on $J\backslash H$. See [**C-M**], [**Wol1**] and [**Kob**]. The second basic problem is to characterize all Γ for which M is a compact manifolds (cocompact lattices). There are examples in which cocompact lattices do not exist [**B-L**] and [**Zim2**]. The most known problem in this area is the Auslander's conjecture. Here $H = Aff(\mathbf{R}^n)$ - the Lie group of all affine transformations of $\mathbf{R}^n$, $J = GL(n, \mathbf{R})$. Then Auslander's conjecture claims that if M is compact then Γ is virtually solvable. See [**Mil**], [**Mar1-2**], [**G-K**], [**Tom**] and [**D-G**].

The aim of this paper to consider the following specific problems of the above type. Let $H = GL(n, \mathbf{R}), J = GL(m, \mathbf{R}), 1 < m < n$, where J is standardly embedded in the upper left corner in H. Note that $J\backslash H = SL(m, \mathbf{R})\backslash SL(n, \mathbf{R})$. We characterize all discrete subgroups $\Gamma \subset GL(n, \mathbf{R})$ which act properly discontinously on $J\backslash H$. Let $M = J\backslash H/\Gamma$ be a locally homogeneous manifold. We then show that $GL(n-m, \mathbf{R})$, standardly embedded in the lower right corner of $GL(n, \mathbf{R})$, acts naturally from the left on $J\backslash H$ and this action projects to M. In particular, any one parameter subgroup of $G(n-m, \mathbf{R})$ induces a flow on M.

In [**Zim2**] it was shown that for $2 \le m < \frac{n}{2}, n > 4$ $J\backslash H$ do not admit cocompact lattices. We make the following observation. Assume that $M = J\backslash H/\Gamma$ is compact. Then Γ, considered as a group of transformation on certain compactification of $J\backslash H$, does not have an invariant probability measure. It was our plan to prove the existence of such an invariant probability measure using the flow induced by the action of corresponding subgroups of $GL(n-m, \mathbf{R})$ and the compactness of M (some generalized version of the Raghunathan's measure conjecture [**Rat**]) but we failed to do so.

We now outline briefly the contents of the paper. In §2 we show that $GL(m, \mathbf{R})\backslash GL(n, \mathbf{R})$ is essentially the Grassmanian G_{mn} times $M^0_{(n-m)n}$ - the set of all $(n-m) \times n$ matrices of rank $n-m$. We then give necessary and sufficient conditions for a discrete group $\Gamma \subset GL(n,\mathbf{R})$ to act properly discontinously (from the right) on M^0_{kn}. In Section 3 we characterize discrete groups $\Gamma \subset GL(n,\mathbf{R})$ which act properly discontinuously on $G(m,\mathbf{R})\backslash G(n,\mathbf{R})$. §4 characterizes subgroups $\Gamma \subset GL(n,\mathbf{R})$, considered as groups of continuous transformations of S_{kn}, which have an invariant probability measure. In the last section we show that if $M = GL(m,\mathbf{R})\backslash GL(n,\mathbf{R})/\Gamma$ is a compact manifold then Γ, considered as a group of continuous transformations of $G_{mn} \times S_{(n-m)m}$, does not have an invariant probability measure on $G_{mn} \times S_{(n-m)m}$.

§2. The action of discrete groups on the frame spaces

Let M_{mn} be the space of $m \times n$ real valued matrices. We view each $X = (x^i_j)^{i=m,j=n}_{i=j=1}$ as composed of m-row vectors $x^i = (x^i_1, ..., x^i_n) \in \mathbf{R}^n, i = 1, ..., m$. Recall that for any $1 \le k \le \min(m,n)$ the matrix $C_k(X)$, the $k-th$ compound of X, is an $\binom{m}{k} \times \binom{n}{k}$ matrix whose entries are $k \times k$ minors of X arranged in a lexigraphical order. See for example [**Gan**] for the standard properties of compound matrices. In what follows we assume that $1 \le m < n$ unless stated otherwise. Then $C_m(X) \in \mathbf{R}^N, N = \binom{n}{m}$. In that case $C_m(X)$ is identified with the wedge product $x^1 \wedge \cdots \wedge x^m$ of the m rows of X. Clearly, $C_m(X) = 0 \iff rank(X) < m$. Let M^0_{mn} be the manifold of $m \times n$ matrices of rank m. This manifold is called the manifold of m- frames in $\mathbf{R}^n$. It is known that for $X, Y \in M^0_{mn}$ $C(X), C(Y)$ are proportional iff the row spaces of the matrices X, Y are identical, e.g. [**Boo**, p'65]. Let $\pi : \mathbf{R}^N\backslash\{0\} \to \mathbf{RP}^{N-1}$ be the canonical projection. Then $\pi(C_m(M_{mn}))$ is the Grassmanian G_{mn}. Set $\phi : M^0_{mn} \to G_{mn}$ to be given by $\phi(X) = \pi(C_m(X))$. Let $X \in M_{mn}, Y \in M_{(n-m)n}$. Note that $C_m(X), C_{n-m}(Y) \in \mathbf{R}^N, N = \binom{n}{m}$. Let $A(X,Y) \in M_{nn}$ be the matrix whose first m rows is the matrix X and and the last $n-m$ rows is the matrix Y. Expanding $det(A(X,Y))$ by the first m row we deduce that

$$det(A(X,Y)) = < C_m(X), C_{n-m}(Y) > .$$

It is not difficult to see that for $u, v \in \mathbf{R}^N$ we can define a "product" $< u, v >$ such that the above equality holds. Moreover, $< v, u >= (-1)^{m(n-m)} < u, v >$.

Let $S \subset M_{nn}$ be the subvariety of singular matrices. That is, $GL(n,\mathbf{R}) = M_{nn}\backslash S$. We view $M^0_{mn} \times M^0_{(n-m)n}$ as a subset of M_{nn}. Set $S' = S \cap M^0_{mn} \times M^0_{(n-m)n}$. Thus, $GL(n,\mathbf{R}) \equiv M^0_{mn} \times M^0_{(n-m)n}\backslash S'$. Note that S' is the variety of "perpendicular" matrices

$$S' = \{A(X,Y) :\ X \in M^0_{mn}, Y \in M^0_{(n-m)n}, < C_m(X), C_{n-m}(Y) >= 0\}.$$

Let

$$\psi : M^0_{mn} \times M^0_{(n-m)n} \to G_{mn} \times M^0_{(n-m)m},\ \psi((X,Y)) = (\phi(X), Y).$$

Set

$$\psi(GL(n,\mathbf{R})) = \psi(M^0_{mn} \times M^0_{(n-m)m}\backslash S') =$$
$$T_{mn} = \{(Z,Y),\ Z \in G_{mn}, Y \in M^0_{(n-m)n}, < Z, Y >\neq 0\}.$$