A List of Index Reduction Formulars

by A.S. Merkurjev, I.A. Panin, and A.R. Wadsworth

Let X be a projective variety over the field F. Let G be a connected semisimple adjoint linear algebraic group which acts on X over F such that for the points over the separable closure Fs of $F, G(Fs)$ acts transitively on $X(Fs)$. In [P], the second author computed the Quillen K-theory $K_*(X; D)$ for any such X and any central simple algebra D over F. This leads by the method of Schofield and van den Bergh in [SB, Lemma 1.4] to an index reduction formula, i.e., a formula for the Schur index

$$\operatorname{ind}(D \otimes_F F(X)),$$

where $F(X)$ is the function field of the variety X. The value of $\operatorname{ind}(D \otimes_F F(X))$ is expressed in terms of certain algebras canonically associated to G and the dimensions of certain representations of a parabolic subgroup of the simply-connected cover of G. In [MPW], we have determined the algebras and computed the integers appearing in the general formula from the varieties associated to all the simple algebraic groups G and for products and transfers of such varieties. We give here a list summarizing the resulting explicit index reduction formulas in all cases where G is one of the classical groups, except for the "trialitarian" groups of outer type D_4.

The index reduction formula given here allow one in particular to complete easily the kernel of the Brauer group map $\operatorname{Br}(F) \to \operatorname{Br}(F(X))$. These kernels were determined in a different way, without using K-theory, in [MT].

The function fields $F(X)$ considered here are treated in [KR] from a different perspective; they are generic for the property of G having a parabolic subgroup in a certain conjugacy class defined over the field.

We use the following terminology and notation: If A is a central simple F-algebra, then the *degree* of A is $\deg(A) = \sqrt{\dim_F(A)}$. By Wedderburn's Theorem, $A \cong Ms(B)$ ($= s \times s$ matrices over B) where B is a division algebra. Then, the *Schur index* of A is $\operatorname{ind}(A) = \deg(B) = \deg(A)/s$. Let $A^{\otimes j} = A \otimes_F A \otimes_F \ldots \otimes_F A (j \text{ times })$. Set $\exp(A)$ to be the order of the class of A in the Brauer group $\operatorname{Br}(F)$. Let A^* be the multiplicative group of units of A. We abuse notation by describing varieties in terms of their F-points, where it is to be understood that for fields $L \supseteq F$, the L-points are obtained in the corresponding way after scalar extension. For example, if A

is a central simple F-algebra of degree n, then the Brauer-Severi of A is $B8 = \{I \mid I$ is a right ideal of F with $\dim_F(I) = n\}$ (viewed as a closed subvariety of the projective G rassmannian variety of n-dimensional F-vector subspaces of the n^2-dimensional vector space A). By this we mean, more precisely, that $B8$ is the variety with L-points $B8(L) = \{J \mid J$ is a right ideal of $A \otimes_F L$ with $\dim_L(J) = n\}$.

The 2-adic valuation $v_2 : \mathbb{Z} \to \mathbb{Z}$ will appear in several formulas below. For $n \in \mathbb{Z} n \neq 0$ set $v_2(n) = a$ if $n = 2^a b$ with b odd.

For the terminology associated to quadratic forms and algebras the involution, Scharlau's book [Sch] is a good reference.

We list the index reduction formulas according to the type of the Dynkin diagram of the associated algebraic group (cf. [T1]), Throughout D denotes any central simple algebra over F.

I. Type A_{n-1}, inner type

Let A be any central F-algebra of degree $n, n \geq 2$, and let $G = PGL(A) = A^*/F^*$. For integers $n_i, 1 \leq n_1 < n_2 < \cdots < n_k \leq n-1$, set

$$X(A; n_1, n_2, \ldots, n_k) = \Big\{ (I_1, I_2, \ldots, I_k) \mid I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k, \text{ each } I_j \text{ is a right ideal of } A, \text{ and } \dim_F(I_j) = n \cdot n_j \Big\}.$$

Set $d = \gcd(n_1, \ldots, n_k, n)$. Then,

$$\operatorname{ind}\Big(D \otimes_F F(X(A; n_1, \ldots, n_k))\Big) = \gcd_{1 \leq i \leq n} \left(\frac{d}{\gcd(i,d)} \operatorname{ind}\left(D \otimes_F A^{\otimes i}\right) \right).$$

Special cases: (a) If $k = 1, n_1 = 1$, then $X(A; 1)$ is the Brauer-Severi variety of A, and the index reduction formula coincides with the one given in [SB1] by Schofield and van den Bergh (here $d = 1$, so each $d/\gcd(i,d) = 1$).

(b)If $k = 1, n_1 = r, 1 \leq r \leq n$, then $X(A; n)$ is the generic partial splitting variety studied by Blanchet [B] and Schofield and van den Bergh [SB2]. The index reduction formula given here for $F(X(A; n))$ is equivalent to the more complicated one given in [B], and is the same as the one given in [S] and [W].

II. Type A_{n-1}, outer type

Let L be a Galois extension field of F with $[L : F] = 2$, and let τ be the nonidentity F-automorphism of L. Let (A, σ) be a central simple F-algebra A of degree n over $F, n \geq 2$, with an L/F involution σ of the second kind on A. That is, σ is an involution of A with $\sigma|_L = \tau$. Here, $G = \{a \in A^* \mid \sigma(a)a \in F^*\}/L^*$, an algebraic group over F.

When n is even $n = 2m$ there is a *discriminant algebra* $\mathcal{D}(A, \sigma)$ of (A, σ). This $\mathcal{D}(A, \sigma)$ is a central simple F-algebra of degree $\binom{2m}{m}$, exponent 2, and index 1,2, or 4, and is obtainable as follows (cf. [M2,] or