ON INTEGRABILITY IN THE LIL FOR DEGENERATE U–STATISTICS

BY EVARIST GINÉ*

University of Connecticut

Runninghead: LIL for U-statistics.

Abstract

It is shown by example that square integrability of the kernel h of a completely degenerate U–statistic is not a necessary condition for the law of the iterated logarithm to hold. h must however enjoy some integrability. Sufficient conditions for a degenerate U–statistic of order 2 to satisfy the LIL, weaker than square integrability of the kernel, are also given.

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1. Introduction. Recently Delhing (1989) and Arcones and Giné (1993b) proved that, for *completely degenerate* kernels, if $Eh^2 < \infty$ then the law of the iterated logarithm

$$\limsup_{n\to\infty} \frac{1}{(2n\log\log n)^{\frac{m}{2}}} \Big| \sum_{(i_1,\ldots,i_m)\in I_n^m} h(X_{i_1},\ldots,X_{i_m}) \Big| < \infty \quad \text{a.s.} \tag{1.1}$$

holds and that, moreover, the limit set of the sequence without the absolute values is, almost surely,

$$K_m(h) = \{Eh(X_1,\ldots,X_m)g(X_1)\cdots g(X_m) : Eg(X_1)^2 \le 1\}. \tag{1.2}$$

Here the notation is standard: X, X_i are independent identically distributed $(S,\mathcal{S})$ valued random variables; $h : S^m \to \mathbf{R}$ is a measurable function invariant under permutations of the coordinates (a *symmetric* kernel) and satisfies

$$Eh(X, x_2, \ldots, x_m) = 0$$

almost surely for $\mathcal{L}(X)^{m-1}$ (a *completely degenerate* or canonical kernel for the law of X); and I_n^m is the set of all m–dimensional vectors whose coordinates form a set of m *different* natural numbers not larger than n. In all that follows we use the notation $\log x$ to really mean $1 \vee \log x$.

If $m \ge 2$ the set K_m may be a bounded set and yet $Eh^2 = \infty$. This indicates that, for $m \ge 2$, the condition $Eh^2 < \infty$ may not be necessary for the LIL (although it is, of course, for $m = 1$). Another indication that this may be the case is the Borell type exponential bound for Rademacher chaos in the line, Theorem 4.11 in Ledoux and Talagrand (1991) with $B = \mathbb{R}$ (see (2.10) below). This inequality implies that the bounded LIL for

$$\sum_{I_n^2} \varepsilon_i\varepsilon_j h(X_i, X_j)$$

reduces to the following two limits

$$\limsup_n \frac{1}{n^2\log\log n} \sum_{I_n^2} h^2(X_i, X_j) < \infty \text{ a.s.}$$

and

$$\limsup_n \frac{1}{n} \sup_{|u|\le 1} \Big| \sum_{I_n^2} u_i u_j h(X_i, X_j) \Big| < \infty \text{ a.s.}$$

(where $u = (u_1, \ldots, u_n)$ and $|u|^2 = \sum_{i=1}^n u_i^2$), none of which requires $Eh^2 < \infty$. In fact the first holds if $E(h^2/\log\log|h|) < \infty$, even with a.s. limit zero. But we do not know how to handle the second in general.

The main object of this note is to present an example of a degenerate kernel h of two variables such that $Eh^2 = \infty$, actually, $E(h^2/\log\log|h|) = \infty$, but such that the corresponding U–statistic satisfies the LIL. In order to avoid complications the case $m > 2$