Regularity of invariant measures on finite and infinite dimensional spaces and applications

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Abstract

In this paper we prove new results on the regularity (i.e., smoothness) of measures μ solving the equation $L^*\mu = 0$ for operators of type $L = \Delta + B \cdot \nabla$ on finite and infinite dimensional state spaces E. In particular, we settle a conjecture of I. Shigekawa in the situation where $\Delta = \Delta_H$ is the Gross-Laplacian, (E, H, γ) is an abstract Wiener space and $B = -\mathrm{id}_E + v$ where v takes values in the Cameron-Martin space H. Using Gross' logarithmic Sobolev-inequality in an essential way we show that μ is always absolutely continuous w.r.t. the Gaussian measure γ and that the square root of the density is in the Malliavin test function space of order 1 in $L^2(\gamma)$. Furthermore, we discuss applications to infinite dimensional stochastic differential equations and prove some new existence results for $L^*\mu = 0$. These include results on the "inverse problem", i.e., we give conditions ensuring that B is the (vector) logarithmic derivative of a measure. We also prove necessary and sufficient conditions for μ to be symmetrizing (i.e., L is symmetric on $L^2(\mu)$). Finally, a substantial part of this work is devoted to the uniqueness of symmetrizing measures for L. We characterize the cases, where we have uniqueness, by the irreducibility of the associated (classical) Dirichlet forms.

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0 Introduction

The topic of this paper is the study of elliptic equations for measures on infinite dimensional spaces. More precisely, we discuss equations of the following type:

$$L^*\mu = 0, \tag{0.1}$$

where μ is a (finite) measure on a locally convex space X, and the operator L is defined on smooth finitely based functions by

$$Lf = \Delta_H f + {}_{X^*}\langle f', B\rangle_X. \tag{0.2}$$

Here H is a fixed Hilbert space continuously and densely embedded into X, and Δ_H is Gross's Laplacian [40], generated by H. $B: X \to X$ is a Borel map such that

$${}_{X^*}\langle \ell, B(\cdot)\rangle_X \in L^2(\mu) \quad \text{for all} \quad \ell \in X^* . \tag{0.3}$$

A measure μ is said to be a solution of (0.1), if for all smooth finitely based functions f

$$\int\limits_X Lf(x)\,\mu(dx) = 0. \tag{0.4}$$

The growing interest in such equations is motivated by numerous applications in mathematical physics, in particular, in quantum field theory, and in infinite dimensional stochastic analysis. A closely related problem (which we also discuss in this paper in Section 4 below), is the study of invariant and symmetrizing measures for infinite dimensional diffusions, more precisely processes $(\xi_t)_{t\geq 0}$ which are weak solutions of stochastic differential equations of type

$$d\xi_t = \sqrt{2}dw_t^H + B(\xi_t)\,dt. \tag{0.5}$$