Circular Coloring of Weighted Graphs

W. Deuber and X. Zhu

July 9, 1994

Abstract

Suppose that G is a finite simple graph and w is a weight function which assigns to each vertex of G a nonnegative real number. Let C be a circle of length t. A t-circular coloring of (G, w) is a mapping Δ of the vertices of G to arcs of C such that $\Delta(x) \cap \Delta(y) = \emptyset$ if $xy \in E(G)$ and $\Delta(x)$ has length $w(x)$. The circular-chromatic number of (G, w) is the least t for which there is a t-circular coloring of (G, w). This paper discusses basic properties of circular chromatic number of a weighted graph and relations between this parameter and other graph parameters. We are particularly interested in graphs G for which the circular-chromatic number of (G, w) is equal to the fractonal clique weight of (G, w) for arbirtary weight function w. We call such graphs star-superperfect. We prove that odd cycles and the complement of odd cycles are star-superperfect. We then consider circular chromatic number of lexicographic product of graphs, which provides a tool of constructing new star-superperfect graphs from old ones.

1 Introduction

Let $G = (V, E)$ be a graph, and $w : V \to [0, \infty)$ be a weight function. Let C be a circle of length t in the Euclidean space R^2. A *t-circular coloring* of (G, w) is a mapping Δ of the vertices of G to open arc of C such that

1. if $(x, y) \in E$ then $\Delta(x)$ and $\Delta(y)$ are disjoint; and

2. for all vertices $x \in V$ the length of the arc $\Delta(x)$ is at least $w(x)$.

The *circular-chromatic number* $\chi_c(G, w)$ of (G, w) is defined as

$$\chi_c(G, w) = inf\{t : \text{there is a } \ t - \text{circular coloring of } (G, w)\}.$$

The circular coloring and circular chromatic number of weighted graphs can be used to model traffic control systems at traffic intersections. Each vertex of the graph represents a traffic flow, and two vertices are adjacent if the corresponding traffic flows are incompatible. The weight of a vertex may be proportional to the amount of traffic in the corresponding traffic flow. The traffic control system assigns to each traffic flow an interval of time in which the light is green. The pattern of the red-green lights will be repeated periodically. In this setting, the circle C then represents a whole period, and the arc assigned to a vertex represents the time interval in which the light is green for the corresponding traffic. As adjacent vertices are assigned to disjoint arcs, non-compatible traffic flows will not have green lights at the same time. It is then natural that we would like to minimize the total length of the period, that is the total length of the circle, subject to the condition that each vertex x is assigned an arc of pre-assigned length, $w(x)$. Circular graphs were used to model traffic control before, (cf. [5]). However, different objectives are approached there.

The parameter χ_c is closely related to many other graph parameters. First we observe that $\chi_c(G, w)$ can be regarded as a generalization of the star-chromatic number $\chi^*(G)$ to weighted graphs.

The star-chromatic number of a simple graph G is a natural generalization of the chromatic number of graph. It was first introduced by A. Vince in 1988, [8]. For integers $k \geq 2d$, a (k, d)-coloring of a graph G is a coloring Δ of the vertices of G with k colors $\{0, 1, 2, \cdots, k-1\}$ so that for any two adjacent vertices x, y we have $d \leq |\Delta(x) - \Delta(y)| \leq k - d$. The ordinary k-coloring is just $(k, 1)$-coloring of G. The star-chromatic number $\chi^*(G)$ of G is the minimum of the ratios k/d for which there is a (k, d)-coloring of G.

A result in [10] can be restated as that $\chi^*(G) = \chi_c(G, w)$, where w is the weight function which takes constant value 1. In this paper, we discuss some basic properties of the circular chromatic number of a weighted graph.