On Joint Eigenvalues of Commuting Matrices

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September 28, 1994

Abstract

A spectral radius formula for commuting tuples of operators has been proved in recent years. We obtain an analog for all the joint eigenvalues of a commuting tuple of matrices. For a single matrix this reduces to an old result of Yamamoto.

1 Introduction, formulation of the result

Let $T = (T_1, \ldots, T_s)$ be an s-tuple of complex $d \times d$-matrices. The *joint spectrum* $\sigma_{pt}(T)$ is the set of all points $\lambda = (\lambda_1, \ldots, \lambda_s) \in C^s$ (called *joint eigenvalues*) for which there exists a nonzero vector $x \in C^d$ (called *joint eigenvector*) satisfying

$$T_j x = \lambda_j x \text{ for } j = 1, \ldots, s. \tag{1}$$

If the $T_i's$ are commuting then $\sigma_{pt}(T) \neq \emptyset$. The joint spectrum can be read off the diagonal of the common triangular form: There exists a unitary $d \times d$ - matrix U such that

$$U^H T_j U = \begin{pmatrix} \lambda_1^{(j)} & \ldots & \ldots & \ldots \\ 0 & \lambda_2^{(j)} & \ldots & \ldots \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_d^{(j)} \end{pmatrix} \text{ for } j = 1, \ldots, s. \tag{2}$$

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Then

$$\sigma_{pt}(T) = \{\lambda_i = (\lambda_i^{(1)}, \ldots, \lambda_i^{(s)}) : i = 1, \ldots, d\}.$$

We order the joint eigenvalues according to their norms

$$||\lambda_1|| \geq \ldots \geq ||\lambda_d||. \tag{3}$$

Here $||.||$ denotes the Euclidean norm in C^r and will later on also denote the associated operator norm for matrices. We omit the reference to the dimensions.

The s-tuple T can be identified with a linear operator mapping C^d into C^{sd}. If $S = (S_1, \ldots, S_m)$ is another m-tuple of $d \times d$-matrices, we define as TS the $sm-$tuple of matrices, whose entries are $T_i S_j, i = 1, \ldots, s, j = 1, \ldots m$, ordered lexicographically. Continuing in this way we define T^m, consisting of s^m entries, each of which is a product of m of the $T_i's$. Identifying again T^m with an operator mapping C^d into $C^{s^m d}$, T^m has d singular values

$$s_1(T^m) \geq s_2(T^m) \geq \ldots \geq s_d(T^m). \tag{4}$$

In this note we will prove

Theorem 1 *For any s-tuple $T = (T_1, \ldots, T_s)$ of commuting $d \times d$-matrices*

$$lim_{m \to \infty}(s_j(T^m))^{\frac{1}{m}} = ||\lambda_j|| \qquad j = 1, \ldots, d. \tag{5}$$

For $j = 1$ this has been proved in [2] , hence we know

$$||\lambda_1|| = lim_{m \to \infty}(s_1(T^m))^{\frac{1}{m}}. \tag{6}$$

We also remark that (6) has been proved in [1] for l_p-norms and in [5] for infinite-dimensional Hilbert spaces. If $s = 1$ then T^m is the usual m-th power of $T = T_1$, and the joint spectrum is the usual spectrum. For this case (5) has been proved by Yamamoto [6], who showed that for a $d \times d-$matrix T with eigenvalues λ_i ordered according to their moduli

$$lim_{m \to \infty}(s_j(T^m))^{\frac{1}{m}} = |\lambda_j| \qquad j = 1, \ldots, d. \tag{7}$$

We will prove Theorem 1 in the following section.