General edge–isoperimetric inequalities

Part I.
Informationtheoretical methods

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1 Introduction

In Combinatorics we often meet two kinds of extremal problems. In one kind optimal configurations consist of "objects", which are somehow uniformly *spread* in the space under consideration, and in the other kind optimal configurations consist of "objects", which are some how *compressed*. To the first kind belong packing, covering and coding problems, whereas diametric (especially of Erdös–Ko–Rado–type), vertex– and edge–isoperimetric problems belong to the second kind.

For many problems of the spreading type the probabilistic method gives good or even asymptotically optimal results, but mostly strictly optimal configurations are unknown. In contrast, problems of compressing type can often be solved exactly with pushing techniques ("pushing down", "pushing to the left" etc., see [14]). However, the success of pushing operations is linked to the property that there is a "nested" structure of optimal configurations with respect to some order. When this is not the case, then there are competing configurations (for example in [6]) and solutions harder to obtain.

We concentrate here on edge–isoperimetric problems. They can be defined for any graph $G = (\mathcal{V}, \mathcal{E})$ as follows. For any $A \subset \mathcal{V}$ define the set $\mathcal{B}(A)$ of all boundary edges, that is,

$$\mathcal{B}(A) = \big\{\{x,y\} \in \mathcal{E} : |\{x,y\} \cap A| = 1\big\}. \tag{1.1}$$

Problem 1:

For given positive integer m find a set $A \subset \mathcal{V}$ of cardinality m with minimal possible value of $|\mathcal{B}(A)|$.

A similar problem is this.

Problem 2:

For given positive integer m find a set $A \subset \mathcal{V}$ of cardinality m with maximal possible value of $|\mathcal{J}(A)|$, where $\mathcal{J}(A) = \big\{\{x,y\} \in \mathcal{E} : \{x,y\} \subset A\big\}$ is the set of inner edges of A.

Notice that for regular graphs G of degree d

$$|\mathcal{B}(A)| + 2|\mathcal{J}(A)| = d|A|$$

and in this case Problems 1 and 2 are equivalent in the sense that a solution of one of these problems is at the same time a solution of the other.

Most results in the literature concern graphs whose vertex set $\mathcal{V}$ is a cartesian product $\mathcal{X}^n = \prod_{t=1}^{n} \mathcal{X}_t$ of sets $\mathcal{X}_t = \{0, 1, \dots, \alpha_t\}$ and whose edges are pairs of vertices with distance 1 under a specified metric ρ.

For the Hamming metric Problems 1, 2 were first solved in the binary case (i.e. when $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$) by Harper [16] and for arbitrary finite α_t's by Lindsey [20]. (The results have been rediscovered many times: [8], [11],...,[18]). They proved that for