Regularity of invariant measures for a class of perturbed Ornstein–Uhlenbeck operators

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Abstract

In the framework of [5] we prove regularity of invariant measures μ for a class of Ornstein–Uhlenbeck operators perturbed by a drift which is not necessarily bounded or Lipschitz continuous. Regularity here means that μ is absolutely continuous with respect to the Gaussian invariant measure of the unperturbed operator with the square root of the Radon–Nikodym density in the corresponding Sobolev space of order 1.

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1 Introduction and main result

The purpose of this paper is to apply the main result in [3], see also [6], and a modification of a technique from [2] to prove regularity of invariant measures

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in the cases considered in [5] but without assuming that the non–linear part of the drift is bounded or Lipschitz continuous.

Consider the stochastic equation

$$\begin{cases} dX = (AX + F(X))dt + dW, \\ \\ X(0) = x \in H \end{cases} \tag{1}$$

on a real separable Hilbert space H, with inner product $< \cdot, \cdot >$ and norm $|\cdot|$. We assume throughout this paper that A is a non negative definite self–adjoint operator on H such that A^{-1} is trace–class. Let $S(t) := e^{tA},\ t \geq 0$. For $F = 0$ it is known that (1) has a well defined H–valued mild solution given by

$$X(t) = S(t)x + \int_0^t S(t-s)dW(s), \tag{2}$$

which has as unique invariant (probability) measure the Gaussian measure $\mu_0 = \mathcal{N}(0, \frac{1}{2}A^{-1})$ (cf.[4]).

Suppose that the following condition holds

$$F \ \text{ is bounded and Lipschitz continuous.} \tag{3}$$

Then the integral version of (1), i.e.

$$X(t) = S(t)x + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)dW(s), t \geq 0, \tag{4}$$

has a unique solution $X(t,x),\ t \geq 0$,(cf.[4]). It also follows (cf.[5, Theorem 3.5]) that this solution has a unique invariant measure μ_F, μ_F is absolutely continuous with respect to μ_0 and

$$\frac{d\mu_F}{d\mu_0} \in W^{1,2}(H, \mu_0).$$

Here the Sobolev space $W^{1,2}(H,\mu_0)$ is defined as follows: Let $\mathcal{F}C_b^\infty(H)$ be the set of all functions φ on H of the form

$$\varphi(x) = f(< e_1, x >, ..., < e_n, x >),\ x \in H,$$

where $n \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^n)$, and $\{e_j |\ j \in \mathbb{N}\}$ is an eigenbasis of A which we *fix from now on.* For $\varphi \in \mathcal{F}C_b^\infty(H)$ let φ' denote its Fréchet derivative, and for $x \in H$ let $D_x\varphi(x)$ denote the element in H corresponding to $\varphi'(x) \in H^*$. Then $W^{1,2}(H,\mu_0)$ is defined as the completion of $\mathcal{F}C_b^\infty(H)$ with respect to the norm given by

$$||\varphi||^2_{W^{1,2}(H,\mu_0)} = ||\varphi||^2_{L^2(H,\mu_0)} + \int_H |D_x\varphi(x)|^2 \mu_0(dx). \tag{5}$$