SOME ALGEBRAICALLY COMPACT MODULES. I

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Abstract. Given a finite dimensional monomial algebra, one knows that some finite dimensional indecomposable modules may be described by words (finite sequences of letters) using as letters the arrows of the quiver and their formal inverses. To every word w, one can attach a so-called string module $M(w)$. Here, we are going to construct certain infinite dimensional modules: We will consider $\mathbb{N}$-words and $\mathbb{Z}$-words (thus infinite sequences of letters) satisfying suitable periodicity conditions. To every such $\mathbb{N}$-word or $\mathbb{Z}$-word x, we describe an algebraically compact module $C(x)$. This module $C(x)$ is obtained from the corresponding string module $M(x)$ as a kind of completion.

Keywords. Algebraically compact modules, pure injective modules. Linearly compact modules. Quivers. Words, $\mathbb{N}$-words, $\mathbb{Z}$-words. String modules, Prüfer modules, $\mathbf{p}$-adic modules. Special biserial algebras. Kronecker modules. Shift endomorphism.

1. Finite Words and Finite Dimensional Modules

Let k be a field. Let A be a finite dimensional monomial algebra, thus $A = kQ/\mathcal{I}$ where $Q = (Q_0, Q_1)$ is a finite quiver and $\mathcal{I}$ an admissible ideal generated by monomials. (We recall that an *admissable ideal generated by monomials* is an ideal $\mathcal{I}$ of the path algebra kQ which is generated by paths ("monomials") of length at least 2 such that all paths of length n are contained in $\mathcal{I}$, for some n.)

Given an arrow α with starting point $s(\alpha)$ and terminal point $t(\alpha)$, we denote by α^{-1} a formal inverse of α, with starting point $s(\alpha^{-1}) = t(\alpha)$ and terminal point $t(\alpha^{-1}) = s(\alpha)$. Given such a formal inverse $l = \alpha^{-1}$, one writes $l^{-1} = \alpha$. Let $\overline{Q}_1$ be the set of all arrows and their formal inverses. We consider words using the elements of $\overline{Q}_1$ as letters (the arrows will be said to be *direct* letters, their formal inverses will be said to be *inverse* letters); a *word* of *length* $n \geq 1$ is a sequence of the form

$$w = l_1 l_2 \cdots l_n \qquad \text{with} \qquad s(l_i) = t(l_{i+1}) \qquad \text{for all} \qquad 1 \leq i < n$$

(one may consider w just as the sequence $(l_1, l_2, \ldots, l_n)$, but it will be convenient, to delete the brackets and the colons), and one requires the following:

(W1) We have $l_i^{-1} \neq l_{i+1}$, for all $1 \leq i < n$.
(W2) No proper subsequence of w or its inverse belongs to $\mathcal{I}$.

(By definition, the *inverse* of $w = l_1 \ldots l_n$ is $w^{-1} = l_n^{-1} \ldots l_1^{-1}$; a proper subsequence of w is of the form $l_i l_{i+1} \ldots l_{j-1} l_j$ (with $1 \leq i < j \leq n$; if w is a word, the subsequences of w are called *subwords.* Usually, one also introduces words of length 0, corresponding to the vertices of the quiver; but we will not need them). Given a word $w = l_1 l_2 \cdots l_n$, the vertex $s(w) = s(l_n)$ is called the starting point of w, the vertex $t(w) = t(l_1)$ is called the terminal point of w. (In order to avoid any irritation, let us stress: since we deal with left modules, we write $\beta\alpha$ for the composition of the arrows $\alpha\colon a \to b$ and $\beta\colon b \to c$; thus, given a module M, and a word $w = l_1 l_2 \cdots l_n$, the subspace wM of M will be obtained by first applying l_n to M, then l_{n-1}, and so on, and finally l_1. Nevertheless we usually will index the letters of the word $w = l_1 l_2 \cdots l_n$ from left to right; and we will call l_1 the *first* letter of w, and l_n the *last* letter.)

Given words $w = l_1 l_2 \cdots l_n$ and $w' = l'_1 l'_2 \cdots l'_m$, the *concatenation* of w and w' is $ww' = l_1 l_2 \cdots l_n l'_1 l'_2 \cdots l'_m$, provided this again is a word (thus we need that $s(w) = t(w')$, that $l_n^{-1} \neq l'_1$, and that also the condition (W2) is satisfied for ww') A word w containing both direct and inverse letters and such that also $w^2 = ww$ is a word is said to be a *cyclic* word. Of course, if w is a cyclic word, we must have $t(w) = s(w)$ (in addition, the first and the last letter of w cannot be inverse to each other, and finally, the condition (W2) has to be satisfied for w^2). Given a cyclic word w, we can form the powers $w^m = w \cdots w$, for $m \geq 1$. The powers w^m with $m \geq 2$ are said to be *proper* powers. The cyclic word w is said to be *primitive* provided it is not a proper power of some other word.

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To every word w, one knows how to attach a finite dimensional indecomposable A-module $M(w)$, or, equivalently, a representation of the quiver Q satisfying suitable relations. Let us recall the construction. Write $w = l_1 l_2 \cdots l_t$, and let $c(i) = s(l_i)$ for $1 \leq i \leq t$ and $c(0) = t(l_1)$. Given two vertices a, b, let

$$k_{ab} = \begin{cases} k & \text{in case } a = b, \\ 0 & \text{otherwise;} \end{cases}$$

in case $a = b$, we denote $e_{ab} = 1_k \in k_{ab}$, otherwise we put $e_{ab} = 0$.

For any vertex a, let

$$M(w)_a = \bigoplus_{i=0}^{t} k_{ac(i)} \ ;$$

an element in $M(w)_a$ is of the form $(\lambda_i)_{0 \leq i \leq t}$ with $\lambda_i \in k_{ac(i)}$. The module $M(w)$ itself is the direct sum of these vector spaces $M(w)_a$, where $a \in Q_0$, thus $M(w) = \bigoplus_{i=0}^{t} k$.

For any arrow $\alpha\colon a \to b$, let $M(w)_\alpha\colon M(w)_a \to M(w)_b$ be defined by

$$M(w)_\alpha\big((\lambda_i)_i\big) = (\mu_i)_i$$

where

$$\begin{aligned} \mu_0 &= \delta(\alpha, l_1)\lambda_1, \\ \mu_i &= \delta(\alpha, l_i^{-1})\lambda_{i-1} + \delta(\alpha, l_{i+1})\lambda_{i+1} \qquad \text{for} \qquad 1 \leq i < t, \\ \mu_t &= \delta(\alpha, l_t^{-1})\lambda_{t-1}; \end{aligned}$$