Indecomposable representations of generalized Weyl algebras

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Abstract

For a class of generalized Weyl algebras which includes the Weyl algebras A_n the tame criteria is given for the problem of describing indecomposable weight and generalized weight modules with supports from a fixed orbit. In tame cases all the indecomposable modules are described.

Introduction

Let an algebra $A = \otimes_1^n \mathcal{A}_i$ be the tensor product over an algebraically closed field K of generalized Weyl algebras $\mathcal{A}_i = \mathcal{D}_i(\sigma_i, a_i)$ of degree 1 (see subsec. 1.1 and 1.4) with basic polynomial rings $\mathcal{D}_i = K[H_i]$ in one variable, where σ_i is any automorphism of $\mathcal{D}_i$ and a_i is any non-zero element of $\mathcal{D}_i$. Then A is a generalized Weyl algebra of degree n. If $a_i = H_i$ and $\sigma_i(H_i) = H_i - 1$ for $i = 1, \dots, n$ the algebra A is isomorphic to the Weyl algebra A_n of degree n (subsec. 1.4). Remark that considered generalized Weyl algebras of degree 1 include the quantum Weyl algebra $A_1(q)$, the quantum plane, the algebra of functions $A(S_q^2)$ on the quantum 2-dimensional sphere and others (see subsec. 1.3).

In the present paper we use the methods of [G], [GP] (see also [Kh], [D2], [DGO]) for study indecomposable weight and generalized weight A-modules. More precisely, we give the criteria when the problem of describing such modules (with support from some fixed orbit) is tame, Theorem 1 and 2, and in the tame cases all such indecomposable modules are classified.

Before the formulation of main results (Theorem 1 and 2) let us give some definitions. On the set of maximal ideals of $\mathcal{D}_i$ resp. $\otimes_{i=1}^n \mathcal{D}_i$ which can be indentified with K resp. K^n acts the group $\mathbf{Z}$ resp. $\mathbf{Z}^n$ by the automorhism σ_i resp. automorphisms $\{\sigma_i\}$. In the first case an orbit $\mathcal{O}_i$ is caled *degenerate* resp. *infinite* if it contains a root of the polynomial a_i resp. infinitely many elements. In the second case an orbit $\mathcal{O}$ is the direct product $\prod_{i=1}^n \mathcal{O}_i$ of orbits $\mathcal{O}_i$. A-module M is *generalized weight* resp. *weight* provided $M = \oplus_{\lambda=(\lambda_1,\dots,\lambda_n)\in K^n} M^\lambda$ resp. $M = \oplus M_\lambda$ where the finite dimensional vector space M^λ resp. M_λ consists of elements $m \in M$ such that $(H_i - \lambda_i)^k m = 0$ resp. $(H_i - \lambda_i)m = 0$ for some $k = k(m)$. The set of all λ such that $M^\lambda \neq 0$ is called the *support* of M. It is easy to prove that support of each (generalized) weight modules belongs to some orbit

$\mathcal{O}$, say. Denote by $\mathcal{W}(\mathcal{O})$ resp. $\mathcal{GW}(\mathcal{O})$ the category of weight resp. generalized weight A-modules with supports from $\mathcal{O}$. One of the aims of the present paper is to prove the following Theorems.

Theorem 1. *1. The problem of describing $\mathcal{W}(\mathcal{O})$ is tame if and only if the number δ of non-degenerate infinite orbits is more or equal to $n-2$, moreover for $\delta = n-2$ all the orbits are infinite and each degenerate orbit contain exactly one root of the polynomial a_i.*

2. If all orbits are infinite and there is no more than one degenerate orbit, then $\mathcal{W}(\mathcal{O})$ contains only finitely many non-isomorphic indecomposable modules.

Theorem 2. *The problem of describing of $\mathcal{GW}(\mathcal{O})$ is tame if and only if $n = 1$, the orbit $\mathcal{O}$ is infinite and contains no more than 2 roots of the polynomial a_1, and if it contains exactly two roots, then both of them are simple, and if one, then its multiplicity is less or equal to 2.*

In the tame cases of Theorem 1 and 2 all the indecomposable modules from the categories $\mathcal{W}(\mathcal{O})$ and $\mathcal{GW}(\mathcal{O})$ are described. Remark, that the category $\mathcal{W}(\mathcal{O})$ is described in [DGO] in the case $n = 1$ and for any commutative ring D.

The plan of the paper is as follows. In sec. 1 all definitions and examples of generalized Weyl algebras are given. Theorems 1 and 2 are proved in sec. 2. In sec. 3 indecomposable weight and generalized weight modules are described in the tame cases. In sec. 4 the results of sec. 2, 3 are used for the classifications of indecomposable weight and generalized weight modules (in tame cases) over the Weyl algebra $A_n(K)$, the quantum plane, the quantum Weyl algebra $A_1(q)$, the factoralgebra $U(\lambda)$ of the universal enveloping algebra $Usl(2)$ of the Lie algebra $sl(2)$, the algebra of functions $A(S_q^2)$ on the quantum 2-dimensional sphere.

1 Generalized Weyl algebras, examples

1.1 Definition of the generalized Weyl algebras ([B1],[B2]).

Let D be a ring, $\sigma = (\sigma_1, ..., \sigma_n)$ a set of commuting automorphisms of D, ($\sigma_i\sigma_j = \sigma_j\sigma_i$, for all i, j), $a = (a_1, ..., a_n)$ be a set of (non-zero) elements of the centre $Z(D)$ of D, such that $\sigma_i(a_j) = a_j$ for all $i \neq j$

The *generalized Weyl algebra* $A = D(\sigma, a)$ (briefly GWA) of degree n with a *basic* ring D is a ring generated by D and the $2n$ indeterminates $X_1^+, ..., X_n^+, X_1^-, ..., X_n^-$ subject to the relations:

$$X_i^- X_i^+ = a_i, \qquad X_i^+ X_i^- = \sigma_i(a_i),$$

$$X_i^{\pm}\alpha = \sigma_i^{\pm 1}(\alpha)X_i^{\pm}, \forall\ \alpha \in D,$$

$$[X_i^-, X_j^-] = [X_i^+, X_j^+] = [X_i^+, X_j^-] = 0,\ \forall\ i \neq j,$$