PRESENTATIONS FOR POWERS OF THE AUGMENTATION IDEAL AND QUADRATIC FORMS

Anthony Bak, Nikolai Vavilov

Fakultät für Mathematik
Universität Bielefeld
33615 Bielefeld, Deutschland

and

Department of Mathematics and Mechanics
University of Sanct-Petersburg
Petrodvorets, 198904, Russia

Let G be a group, $\mathbb{Z}[G]$ its integral group ring and $\eta\colon\ \mathbb{Z}[G] \to \mathbb{Z}$ the augmentation homomorphism. The kernel $J(G)$ of the homomorphism η is called the *augmentation ideal* of $\mathbb{Z}[G]$. It is clear that $J(G)$ is freely generated as an abelian group by the elements

$$[g] := 1 - g, \quad g \in G \setminus \{1\}$$

(cf. [3], [6]). This implies that the n-th power $J^n(G) := J(G)^n$ of the augmentation ideal $J(G)$ is generated as an abelian group by the products

$$[g_1, \dots, g_n] := (1 - g_1) \dots (1 - g_n), \quad g_1, \dots, g_n \in G.$$

Key words and phrases. Integral group ring, powers of the augmentation ideal, 2-cocycle, Witt ring, Pfister classes.

The first named author gratefully acknowledges the hospitality of Prof. E.Tan and the support of the *National Science Council of Taiwan* during the preparation of the initial stages of the current article.

The second named author gratefully acknowledges the support of the *Alexander von Humboldt-Stiftung* during the preparation of the present article.

It is a classical unsolved problem in the theory of group rings to find the relations among the generators $[g_1, \dots, g_n]$ for $J^n(G)$. There are two obvious relations which hold for any group G. The first is the normalizing relation

(N) $$[g_1, \dots, g_n] = 0 \quad (n \geq 1), \text{ whenever some } g_i = 1.$$

The second is that the symbol $[g_1, g_2]$ is a 2-cocycle:

$$[g_2, g_3] - [g_1g_2, g_3] + [g_1, g_2g_3] - [g_1, g_2] = 0.$$

This implies the relation

(R) $$[g_1, \dots, g_n] \quad (n \geq 2) \text{ is a 2-cocycle in } g_{i-1}, g_i$$

when the other variables are fixed. Other relations seem to depend essentially on the subgroup structure of the group G. Even in the case of a finite abelian group, these relations may be quite complicated and are not completely understood.

In the present article, we solve completely this problem in the important special case when G is a group of exponent 2. It turns out that the only relations one has to impose on the symbols $[g_1, \dots, g_n]$, apart from (N) and (R), are the relation

(S) $$[g_1, \dots, g_i, \dots, g_j, \dots, g_n] = [g_1, \dots, g_j, \dots, g_i, \dots, g_n] \qquad (n \geq 2)$$

which follows from the commutativity of G (every group G of exponent 2 is commutative), and the relation

(T) $$[f, g, fg, g_4, \dots, g_n] = 0 \qquad (n \geq 3)$$

which is verified easily by direct computation.

Our main result is the following.

Theorem A. *Let G be a group of exponent 2. Then the group $J^n(G)$ is the free abelian group on the symbols $[g_1, \dots, g_n]$ $(g_i \in G,\ i = 1, \dots, n)$ modulo the relations*

(N),	*if* $n = 1$;
(N), (R), *and* (S),	*if* $n = 2$;
(N), (R), (S), *and* (T),	*if* $n \geq 3$.

In the process of proving Theorem A, we establish the following mod 2 result. Let $\mathbb{F}_2$ denote the field of 2 elements. Let $\mathbb{F}_2[G]$ denote the group ring over $\mathbb{F}_2$, $J_2(G)$ the kernel of the augmentation homomorphism $\eta_2 : \mathbb{F}_2[G] \longrightarrow \mathbb{F}_2$, and $J_2^n(G) := J_2(G)^n$. In the mod 2 situation, the relations (N), (R), and (S) remain unchanged, but the relation (T) is replaced by

(T′) $$[g, g, g_3, \dots, g_n] = 0 \qquad (n \geq 2).$$