Spectral Variation Bounds for Diagonalisable Matrices

Rajendra Bhatia *
Indian Statistical Institute
7, S.J.S. Sansanwal Marg
New Delhi - 110016
India

Ludwig Elsner
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
D-33501 Bielefeld
Germany

Gerd M. Krause
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
D-33501 Bielefeld
Germany

November 15, 1994

Abstract

This note is related to an earlier paper by Bhatia, Davis, and Kittaneh [4]. For matrices similar to Hermitian, we prove an inequality complementary to the one proved in [4, Theorem 3]. We also disprove a conjecture made in [4] about the norm of a commutator.

*This work was done when the first author visited the SFB 343 at University of Bielefeld in May and June 1994

§**1.** Let A be an $n \times n$ (complex) matrix with real eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Let $\mathrm{Eig}^{\downarrow} A$ and $\mathrm{Eig}^{\uparrow} A$ denote the diagonal matrices whose diagonal entries are $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$, respectively. Let $|||\cdot|||$ be any *unitarily invariant norm* on the space of matrices.

If A and B are Hermitian we have the following inequalities that provide lower and upper bounds for $|||A - B|||$ in terms of the distance between their spectra:

$$|||\mathrm{Eig}^{\downarrow} A - \mathrm{Eig}^{\downarrow} B||| \leq |||A - B||| \leq |||\mathrm{Eig}^{\downarrow} A - \mathrm{Eig}^{\uparrow} B|||. \tag{1}$$

See [2, Theorem 9.7]. The first inequality in (1) is a consequence of a theorem of Lidskii and was noted in this form by Mirsky [11], the second inequality is easier to prove and was noted by Sunder [14] and by Bhatia [3].

Now suppose A, B are not necessarily Hermitian but are similar to real diagonal matrices; i.e., there exist invertible matrices S and T such that SAS^{-1} and TBT^{-1} are real diagonal matrices. Let $c(S)$ be the *condition number* of S defined as $c(S) = \|S\| \, \|S^{-1}\|$ where $\|\cdot\|$ is the operator bound norm. Then it was shown in [4] that

$$|||\mathrm{Eig}^{\downarrow} A - \mathrm{Eig}^{\downarrow} B||| \leq c(S) \, c(T) \, |||A - B|||. \tag{2}$$

When A, B are Hermitian, S, T are unitary and the inequality (2) reduces to the first inequality in (1). The corresponding generalisation of the second inequality in (1) is provided below.

Theorem 1 *Let A, B be two matrices such that $A = S\Lambda S^{-1}, B = TMT^{-1}$, where S, T are invertible matrices and Λ, M are real diagonal. Then*

$$|||A - B||| \leq c(S) \, c(T) \, |||\mathrm{Eig}^{\downarrow} A - \mathrm{Eig}^{\uparrow} B|||, \tag{3}$$

for every unitarily invariant norm.

Proof: Note that

$$\begin{aligned} |||A - B||| &= |||S\Lambda S^{-1} - TMT^{-1}||| \\ &= |||S(\Lambda S^{-1} T - S^{-1} TM) T^{-1}||| \\ &\leq \|S\| \, \|T^{-1}\| \, |||\Lambda S^{-1} T - S^{-1} TM|||, \end{aligned} \tag{4}$$

by a familiar property of unitarily invariant norms [2]. So, to prove (3) it would suffice to show that

$$|||\Lambda X - XM||| \leq \|X\| \, |||\Lambda^{\downarrow} - M^{\uparrow}|||, \tag{5}$$