Proximal elements in p-adic groups

Vladimir P. Platonov*

Let K be a field with valuation v. An element $g \in GL_n(K)$ is called proximal if g has a unique eigenvalue λ – counting multiplicities – of maximal value $v(\lambda)$ (We use the same notation v for some extension of v). Proximal elements play an important role in the study of linear groups [T] and of dynamics of linear transformations ([F], [GM], [GR]).

Recently in [GM], [AMS] different properties of proximal elements for $K = \mathbb{R}$ are considered. For example, if G is a simple algebraic group defined over $\mathbb{R}$, such that the group $G(\mathbb{R})$ is irreducible as a linear group and contains a proximal element, then every Zariski dense subgroup $H \subset G(\mathbb{R})$ contains a proximal element, too (in fact, one can find more general results in [AMS]). In this form the analogous result does not hold for p-adic groups even for $K = \mathbb{Q}_p$. For example, $H = SL_n(\mathbb{Z}_p)$ is Zariski dense in $SL_n(\mathbb{Q}_p)$, but $SL_n(\mathbb{Z}_p)$ has no proximal elements. The group $SL_n(\mathbb{Z}_p)$ is compact and therefore bounded in the p-adic topology.

In this connection G. Margulis formulated the following question. Let G be a simple algebraic group defined over $\mathbb{Q}_p$, let $H \subset G(\mathbb{Q}_p)$ be an unbounded Zariski dense subgroup and suppose $G(\mathbb{Q}_p)$ contains a proximal element. Does H contain a proximal element?

We prove in this paper that it is true if either H is a non-discrete subgroup of $G(\mathbb{Q}_p)$ or H is a lattice in $G(\mathbb{Q}_p)$. If H is discrete (but not a lattice) we give the example of Margulis of a subgroup $H \subset SL_4(\mathbb{Q}_p)$ without proximal elements.

We need the following simple lemmas.

Lemma 1 *If $G(\mathbb{Q}_p)$ has a proximal element then the set of proximal elements in $G(\mathbb{Q}_p)$ is open in the p-adic topology.*

Proof. If $g \in G(\mathbb{Q}_p)$ is proximal then for a sufficiently small neighbourhood U of the identity in $G(\mathbb{Q}_p)$ the set gU consists of proximal elements. This follows from the well-known property that the roots of polynomials over $\mathbb{Q}_p$ depend continuously on the coefficients, applied to the characteristic polynomials $\chi_t(x)$ for $t \in gU$.

*Partially supported by NSERC-Grant and Alexander von Humboldt Foundation

Lemma 2 *Let G be a connected algebraic group defined over a complete field K, with non-trivial absolute value, and assume that $G(K)$ is irreducible as a linear group (i.e. the $G(K)$-module K^n is irreducible). If $G(K)$ has a proximal element then $G(K)$ is absolutely irreducible.*

Proof. Let $\overline{K}$ be the algebraic closure of K (we will assume that K is a field of characteristic 0).

If $G(K)$ is not absolutely irreducible then $\overline{K}^n$ is a direct sum of at least two irreducible $G(\overline{K})$-submodules: $\overline{K}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_d$. All V_i are defined over some Galois extension L/K. Then $V_i \simeq V_1^{\sigma_i}$ for suitable $\sigma_i \in \mathrm{Gal}(L/K)$ and the group $G(K)$ is conjugate in $GL_n(L)$ with the group G' having at least two absolutely irreducible blocks, A and A^{σ_2}. Therefore $G(K)$ has no proximal elements.

Lemma 2 reduces the study of proximal elements essentially to the case of an absolutely irreducible $G(K)$.

An algebraic group G is called K-simple if G is defined over K and G has no proper connected normal subgroups defined over K. Of course, any simple group defined over K is K-simple.

Theorem 1 *Let G be a $\mathbb{Q}_p$-simple algebraic group and H be an unbounded Zariski dense non-discrete subgroup of $G(\mathbb{Q}_p)$. If $G(\mathbb{Q}_p)$ has a proximal element then H has a proximal element, too.*

Proof. Let $\overline{H}$ be the p-adic closure of H in $G(\mathbb{Q}_p)$. Then $\overline{H}$ is a p-adic Lie group. Let $L(\overline{H})$ be the Lie algebra of $\overline{H}$ over $\mathbb{Q}_p$. Since $h\overline{H}h^{-1} = \overline{H}$ for any $h \in H$ we have $hL(\overline{H})h^{-1} = L(\overline{H})$. But H is Zariski dense in $G(\mathbb{Q}_p)$, so $g(L(\overline{H}))g^{-1} = L(\overline{H})$ for any $g \in G(\mathbb{Q}_p)$. Then $L(\overline{H})$ is an ideal of $L(G(\mathbb{Q}_p))$ (see [B], p. 347), and, since H is non-discrete, $L(\overline{H}) \neq 0$. The Lie algebra $L(G(\mathbb{Q}_p))$ is simple, therefore $L(G(\mathbb{Q}_p)) = L(\overline{H})$. This means that $\overline{H}$ is an open subgroup of $G(\mathbb{Q}_p)$.

H is an unbounded subgroup of $G(\mathbb{Q}_p)$, therefore $G(\mathbb{Q}_p)$ is not compact and G is isotropic over $\mathbb{Q}_p$. Let $G^+(\mathbb{Q}_p)$ be the subgroup of $G(\mathbb{Q}_p)$ generated by its unipotent elements. It is a non-central normal subgroup, therefore $G(\mathbb{Q}_p)/G^+(\mathbb{Q}_p)$ is finite ([PR], § 3.3). The finiteness of $G(\mathbb{Q}_p)/G^+(\mathbb{Q}_p)$ implies the finiteness of $\overline{H}/G^+(\mathbb{Q}_p) \cap \overline{H}$. Therefore $G^+(\mathbb{Q}_p) \cap \overline{H}$ is an open non-compact subgroup of $G^+(\mathbb{Q}_p)$. Any open unbounded subgroup of $G^+(\mathbb{Q}_p)$ coincides with $G^+(\mathbb{Q}_p)$ by Tits' theorem. Thus, $G^+(\mathbb{Q}_p) \cap \overline{H} = G^+(\mathbb{Q}_p)$, that is $\overline{H} \supseteq G^+(\mathbb{Q}_p)$. If $g \in G(\mathbb{Q}_p)$ is a proximal element then for some $m \in \mathbb{N}^+$ we have $g^m \in G^+(\mathbb{Q}_p)$, and g^m is a proximal element of $\overline{H}$. Then H also has proximal elements by lemma 1.

Theorem 1 is proven.