Classification of integral lattices with large class number

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Abstract

A detailed exposition of Kneser's neighbour method for quadratic lattices over totally real number fields, and of the sub-procedures needed for its implementation, is given. Using an actual computer program which automatically generates representatives for all isomorphism classes in one genus of rational lattices, various results about genera of ℓ-elementary lattices, for small prime level ℓ, are obtained. For instance, the class number of 12-dimensional 7-elementary even lattices of determinant 7^6 is 395; no extremal lattice in the sense of Quebbemann exists. The implementation incorporates as essential parts previous programs of W. Plesken and B. Souvignier.

1 Introduction

We deal with integral lattices L in euclidean spaces, i.e. $L = \mathbb{Z}v_1 + ... + \mathbb{Z}v_n$ consists of the integral linear combinations of a basis $v_1, ..., v_n$ of a real (or rational) vector space V, equipped with a positive definite symmetric bilinear form. Integral means that the form takes integral values on the lattice: $(x, y) \in \mathbb{Z}$ for all $x, y \in L$. Classification of course means classification up to isometry. We refer to [O'Me71], Chap. 19 and [CoSl88a], Chap. 15 for general results about integral lattices (or, equivalently, integral quadratic forms) and their classification, and to [Nie73, Ven80, Que93, SchVe94, Bac93] for more specialized investigations close to the subject of the present paper. We follow the notation used in [SchVe94].

The class number $h(n, d)$ of lattices in dimension n and determinant d tends to infinity rapidly with n and d. This even holds for forms in a specified genus [Pfe71]. Therefore known classifications have been restricted to small values of n or d, or have been subject to further restrictions within one genus of lattices. The aim of this paper is to present a computer program which, for class numbers up to several hundreds and dimensions up to about 16, automatically generates all lattices (in terms of Gram matrices) of a given genus, deletes forms equivalent to previous ones, and finally produces a list of representatives for all isometry classes. In particular, the class number of the genus is obtained. Although

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some of the papers quoted above are partially computer assisted in the sense that certain auxiliary calculations have been carried out with the aid of a computer, a general approach like the present one up to now has been made only in dimension at most 4 [SPi91].

The principal method used for our algorithm is well known: it is Kneser's method of neighbouring lattices, more precisely the theorem of [Kne57] saying that (under mild conditions) every lattice in a genus can be obtained from a given one via a finite sequence of consecutively neighbouring lattices. The implementation of this method in dimensions 3 and 4 given by Schulze–Pillot [SPi91] heavily relies on the existence of a very fast test for isometry, coming from the existence of an almost unique normal form for the Gram matrix with respect to a reduced basis. This no longer holds in higher dimensions. Our main tool then is the –mathematically straightforward– idea of calculating the neighbours of a lattice only up to the action of the orthogonal group of that lattice. Results like those of [SchVe94], or already known tables of quaternary forms indicate that even for large class number it often happens that every lattice in the genus in question has a large orthogonal group. However, actual theorems on the relation between the mass, the class number, the sizes of the orthogonal groups, and the diameter of the neighbour graph on isometry classes are hardly known.

The methods for fast generation of orthogonal groups and for an efficient test for isometry that we use are described in [PlPo85] and [PlSo93]. We use, as subroutines of our program, implementations of the ideas of [PlSo93] kindly provided to us by Bernd Souvignier.

In the third section of this paper, we report on explicit results obtained by our program. We treat certain genera (of determinant $\ell^{n/2}$) containing ℓ-modular lattices, for small prime numbers ℓ, and suitable dimensions n. Thereby we answer some questions about modularity of ℓ-elementary lattices and about the existence and uniqueness of extremal modular lattices which were posed (implicitly) in the work of Quebbemann [Que93]; cf. also [SchVe94], Section 4.

2 The method

In this section we want to explain in some detail how Kneser's method of neighbouring lattices is used in a practical and efficient way to produce, starting from one Gram matrix G (always assumed to be positive definite), Gram matrices $G = G_1, G_2, \ldots, G_h$ which are representatives for all isometry classes in the genus of L. A detailed account of the neighbour method, including a sketch of the underlying theory, has been given recently in [SPi91]. Nevertheless, we found it convenient to recall a few facts here, in particular, concerning the uniqueness of the neighbour defined by a certain congruence class of vectors. At the same time, we take the opportunity to formulate the main facts for the general case of lattices over (totally real) algebraic number fields. The neighbour method has indeed recently been applied to quaternary lattices over real quadratic fields, including cases where the class number of the field is not one, [Hab94]. For the calculation of a basis and thus a Gram matrix of a neighbouring lattice we use a method which for $p \neq 2$ differs from the method proposed in Step 3 of [SPi91]. The main reason why we introduced our