FINITELY GENERATED MODULES OVER PULLBACK RINGS

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Abstract. The purpose of this paper is to outline a new approach to the classification of finitely generated indecomposable modules over certain kinds of pullback rings. If R is the pullback of two hereditary noetherian serial rings over a common semi–simple artinian ring, then this classification can be divided into the classification of indecomposable artinian modules and those modules over the coordinate rings with no non–trivial artinian submodules. The classification of the artinian modules can be reduced to the case of a finite dimensional algebra over a semi–simple ring. This approach is carried out in the case where the coordinate rings are hereditary noetherian serial rings over a common quotient which is a matrix ring over a field.

1. Introduction

A focal point of recent attempts to classify finitely generated indecomposable modules over non–artinian rings has been pullback rings. Notably, there is the monumental work of L. Levy [Le3], resulting in the classification of all finitely generated indecomposable modules over Dedekind-like rings. L. Klingler extended this classification to lattices over certain non–commutative Dedekind-like rings [Kl], and Klingler and J. Haefner [HK1, HK2] classified lattices over certain non–commutative pullback rings, which they called special quasi–triads. Common to all these classifications is the reduction to a "matrix problem" over a division ring which is then solved using techniques similar to those in [KL]. However, as the rings become more complicated, the matrix problems become rather unwieldy, thereby limiting the usefulness of this matrix approach.

At the same time, the theory of finitely generated modules over finite dimensional k–algebras, k a field, has witnessed tremendous progress with the help of new and powerful functorial techniques, like the method of functorial filtrations [CB1-3], and almost split sequences, as well as the theory of bocses, a formalization of matrix methods [CB4]. (See [Ri] for a recent survey). The matrix problems solved in [KL] are special cases of self–reproducing systems [NR], which, in turn, can be interpreted as finitely generated modules over a finite dimensional clannish k–algebra. One of the most sweeping results is Crawley-Boevey's module classification for clannish

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algebras [CB2], which subsumes many earlier results, such as the classifications in [BS, SW, WW]. His functorial filtration techniques are derived from combinatorial arguments of [GP].

There are some intriguing "coincidences" between the clannish algebra situation and the non–artinian rings with local coordinate rings considered by Levy. In both cases, indecomposable modules break up into two classes, called string and band modules in the artin algebra setting, and deleted cycle and block cycle modules in the non–artinian pullback setting. In fact, the deleted cycle and block cycle matrix solutions of [KL] can be described as strings and bands, respectively (see [AL]).

The purpose of this paper is to exploit these coincidences and outline a new approach to the classification of finitely generated indecomposable modules over certain types of pullback rings, including Dedekind–like rings. It remains to be explored how general the class of pullback rings is, to which our method applies.

Specifically, let R_1, R_2 be noetherian rings with a semi–simple artinian ring F as common quotient. Let $v_i : R_i \longrightarrow F,\ i = 1, 2$, be the canonical projections, and let

$$R = \{(r_1, r_2) \in R_1 \times R_2 | v_1(r_1) = v_2(r_2)\}$$

be the pullback ring. If R_1 and R_2 are Dedekind domains, then R is an example of a Dedekind–like ring. In turn, any artinian module over a Dedekind–like ring is a module over a pullback ring with R_1 and R_2 discrete valuation rings [Le3, Prop. 15.1]. Also, if R_1 and R_2 are finite products of hereditary orders over Dedekind domains, then R is an example of a Dedekind–like order [Kl].

For M a finitely generated R–module, define $\mathcal{A}(M)$ to be the unique maximal artinian submodule of M. After deriving some properties of the functor $\mathcal{A}$ in Section 2, and some results on serial rings in Section 3, we will show in Section 4 (Theorem 4.1) that, if R_1 and R_2 are hereditary noetherian serial rings, finitely generated over their centers, and M is indecomposable, then $\mathcal{A}(M)$ is also indecomposable, and M is determined, up to isomorphism, by $\mathcal{A}(M)$ and $M/\mathcal{A}(M)$. For instance, finite products of hereditary orders over complete discrete valuation rings are noetherian serial [Wa, Theorem 6.1]. In this case, $M/\mathcal{A}(M)$ is an R–lattice, hence easily classified up to isomorphism. Thus, classification of indecomposables is reduced to the classification of artinian indecomposables.

In Section 5 we establish a bijection between isomorphism classes of indecomposable R–modules, for a pullback ring R, and isomorphism classes of indecomposable modules over the associated graded ring $G(R)$ of R. The ring $G(R)$ is also a pullback ring.

In Section 6 we derive some properties of $G(R)$. In particular, if R_1 and R_2 are hereditary noetherian serial, and F is a matrix ring over a field k, then $G(R)$ is Morita equivalent to the Gelfand algebra $k[x, y]/\langle xy\rangle$, localized at the maximal ideal $\langle x, y\rangle$, where $k[x, y]$ is the polynomial ring in commuting variables.

Finally, in Section 7, we classify the indecomposable modules over a pullback ring of hereditary noetherian serial rings, with F a matrix ring over a field. This classification uses the complete list of indecomposables in [BS] for the truncations

$$\Lambda_{pq} = k[x, y]_{\langle x,y\rangle}/\langle xy, x^{p+1} - y^{q+1}\rangle.$$

These algebras are classical examples of Crawley-Boevey's clannish algebras, being realized as the path algebra of a quiver with one vertex and two loops x and