Delaunay polytopes of cut lattices

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Abstract

We continue [3], the study of the lattice L_n generated by cuts of the complete graph on a set V_n of n vertices. The lattice L_n spans an $N = \binom{n}{2}$-dimensional space of all functions defined on a set V_n^2 of all unordered pairs of the set V_n. We prove that the cut polytope, i.e. the convex hull of all cuts, is an asymmetric Delaunay polytope of L_n. Symmetric Delaunay polytopes of a lattice L are completely described by classes of the quotient $\frac{1}{2}L/L$. We show that a class of the quotient $\frac{1}{2}L_n/L_n$ is uniquely determined by a subset $S \subseteq V_n$ and a class of switching equivalent sets $A \subseteq V_n^2$. We describe minimal vectors of all classes of $\frac{1}{2}L_n/L_n$. We completely describe L-partition of 6-dimensional space into Delaunay polytopes of the lattice $L_4 = \sqrt{2}D_6^{+2}$.

1 Introduction

Let V_n be a ground set of cardinality $|V_n| = n$. A *cut vector* or simply, a *cut* $\delta(S)$, $S \subseteq V_n$, is a vector of the space of all functions $d : V_n^2 \to \mathbf{R}$, defined on the set V_n^2 of all unordered pairs of the set V_n. A component $\delta_{ij}(S)$ is defined as follows. For $S, T \subseteq V_n$, let

$$(S,T) = \{(ij) \in V_n^2 : \ i \in S, j \in T\}, \ D(S) = (S, V_n - S), \ S^2 = (S,S). \qquad (1)$$

Then

$$\delta_{ij}(S) = \begin{cases} 1 & \text{if } (ij) \in D(S) \\ 0 & \text{otherwise} \end{cases}$$

For any map ϕ defined on subsets of the set V_n and for $i \in V_n$, we abbreviate $\phi(\{i\})$ to $\phi(i)$. In particular, notations $\delta(i)$, $D(i)$ and (i,T) are clear. Note that $(i,j) = \{(ij)\}$.

The dimension of the space spanned by all $\delta(S)$'s is equal to $N := |V_n^2| = \frac{n(n-1)}{2}$. Since $\delta(S) = \delta(V_n - S)$, there are 2^{n-1} cuts, including zero cut $\delta(\emptyset) = 0$.

Let $\mathcal{K}_n = \{\delta(S) : S \subseteq V_{n-1}\}$ be the set of all cut vectors.

Let $\mathbf{Z}^N$ be the lattice of all integral N-vectors. Clearly $\mathcal{K}_n \subset \mathbf{Z}^N$. Hence the lattice L_n linearly generated by all cuts $\delta(S) \in \mathcal{K}_n$, i.e.

$$L_n = \{d : d = \sum_{\delta(S) \in \mathcal{K}_n} z_S \delta(S), \ z_S \in \mathbf{Z}\},$$

is a sublattice of $\mathbf{Z}^N$.

In this note we study an L-partition related to L_n, i.e. the partition of the space $\mathbf{R}^N$ spanned by L_n into Delaunay polytopes. This partition is normal, i.e. it is face-to-face. This means that if two Delaunay polytopes of this partition are contiguous, then they share a common face.

A *Delaunay polytope* of a lattice is a convex hull of all lattice points lying on an empty sphere of the lattice, and the lattice points on the empty sphere have full rank. A sphere is called *empty* if there is no lattice point strictly inside it. A Delaunay polytope P is called *basic* if the set of its vertices contains an affine basis of the lattice affinely generated by vertices of P. A Delaunay polytope P is called *symmetric* or *asymmetric* according to whether an antipode of each vertex of P is or not is (respectively) a vertex of P.

2 Cut polytope is an asymmetric Delaunay polytope of L_n

Let $\{e_{ij} : 1 \le i < j \le n\}$ be an orthonormal basis of $\mathbf{R}^N$. The lattice generated by this basis is $\mathbf{Z}^N$. For $m \in \mathbf{Z}$, let $m\mathbf{Z}^N$ be the lattice of all integral vectors divisible by m. The following simple propositions are proved in [3].

Proposition 1 *(i)* $2\mathbf{Z}^N \subset L_n$,
(ii) $L_n/2\mathbf{Z}^N = \mathcal{K}_n$, *i.e. each* $d \in L_n$ *is equivalent by mod 2 to some* $\delta(S) \in \mathcal{K}_n$.

Proposition 2 *The following set* B *of* N *cuts forms a basis of* L_n

$$B = \{\delta(i), \delta(ij) : i, j \in V_{n-1}\}.$$

Proposition 1(ii) implies that every $d \in L_n$ has the form

$$d = 2a + \delta(S), \text{ where } a \in \mathbf{Z}^N. \tag{2}$$

The equation (2) shows that unique (0,1)-vectors $d \in L_n$ are cut vectors. Let $d \in L_n$ has $(0, \pm 1)$-components. Since d (mod 2) is a cut vector, all such vectors have the following form

$$d(S; X) := \delta(S) - 2 \sum_{(ij) \in X} e_{ij}, \tag{3}$$

where $X \subseteq D(S)$, and the set $D(S)$ is defined in (1). There are $2^{k(n-k)}$, $k = |S|$, subsets $X \subseteq D(S)$.

Since any $X \subseteq D(i)$ has the form $X = (i, T)$ for some $T \subseteq V_n - \{i\}$, if $S = \{i\}$, we simplify notations setting

$$d(i, T) := d(i; (i, T)).$$

For $(ij) \in V_n^2$, let R_{ij} be a reflection of the space $\mathbf{R}^N$ in the hyperplane orthogonal to e_{ij}. Any two reflections R_{ij} and R_{kl} commute, i.e. $R_{ij}R_{kl} = R_{kl}R_{ij}$. Note that for $d(S; X)$ of (3), we have $d(S; X) = \prod_{(ij) \in X} R_{ij}\delta(S)$.

For $T \subseteq V_n - \{i\}$, set $R^i_T = \prod_{j \in T} R_{ij}$. Then $R^i_S R^i_T = R^i_{S \Delta T}$. In this notations, $d(i, T) = R^i_T \delta(i)$.