Pseudo-LYM inequalities and AZ identities

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Abstract

We give pseudo-LYM inequalities in some posets and give a new restriction in this way for their antichains. Typically these posets fail the LYM inequality and some of them are known not to be Sperner. In the latter class our inequality yields an upper bound for the maximum size of an antichain.

1 Introduction

Let us be given a ranked partially ordered set $\mathcal{P}$, in which the set of elements of rank r are denoted by $\mathcal{P}_r$. (Following tradition and natural notation, the smallest rank in a poset is either 0 or 1.) The *profile* of an antichain $\mathcal{A}$ in $\mathcal{P}$ is the sequence of cardinalities $f_r = |\mathcal{A} \cap \mathcal{P}_r|$. The poset $\mathcal{P}$ satisfies the *LYM inequality* ([6], [12], [13] and [17]) if for every antichain $\mathcal{A}$

$$\sum_r \frac{f_r}{|\mathcal{P}_r|} \leq 1.$$

A vast amount of literature investigates which posets have the LYM property. The LYM property has become a central issue due to two facts: (a) the LYM property seems to be the most straightforward mean to prove that a poset has the *Sperner property*, i.e. the largest antichain has size $\max_r |\mathcal{P}_r|$; (b) the LYM property is equivalent to some other, perhaps more illuminating properties of the poset (see e.g. [8]). Recently Ahlswede and Zhang [3] extended the "classical" LYM inequality on the inclusion partial order of the power set into an elegant identity, and LYM inequalities of several other posets has been extended into a corresponding AZ type identity ([1], [2], [4], [16]).

One important poset that fails the LYM inequality is the *partition lattice* of a set ([15]). Contrary to a longstanding conjecture of Rota ([14]), it does not even satisfy the Sperner property ([7]).

The purpose of the present paper is to prove *pseudo-LYM inequalities* for five posets that fail (with one exception) the LYM inequality. The merit of the pseudo-LYM inequality is that it still describes a severe restriction on the profile of an antichain. We also give the AZ identity form of the inequalities and characterize how equality may hold.

Three posets studied in this paper—the partition lattice, the poset of chains of a Boolean algebra, and the poset of chains of subspaces of a finite vector space—all share the property that a poset element can be defined as a set of subsets of a universe. The divisor lattice of a given number and the generalized Boolean algebra exhibit anology to the above phenomenon. This property expectedly implies pseudo-LYM inequalities in other posets as well.

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2 The partition lattice

Let S be an n-element set and let $\mathcal{P}$ be its partition lattice. We say a partition α is less than partition γ iff γ is a refinement of α. For a partition family $\mathcal{A}$ in $\mathcal{C}$ and for $k \geq 1$, $0 < i_1 \leq i_2 \leq \ldots \leq i_k$, let $f_{k;i_1,i_2,\ldots,i_k}(\mathcal{A})$ denote the number of elements of $\mathcal{A}$ of k classes, with class sizes $i_1, i_2, \ldots, i_k$.

Theorem 2.1 *Let $\mathcal{A} \neq \emptyset$ denote an antichain in the partition lattice $\mathcal{P}$ of an $n \geq 2$ element set, i.e. no element of $\mathcal{A}$ is refinement of another. We have*

$$\sum_{\substack{k\geq 1;\\ i_1,i_2,\ldots,i_k}} \frac{f_{k;i_1,i_2,\ldots,i_k}(\mathcal{A})k!}{\binom{n}{i_1, i_2, \ldots, i_k}\binom{n-1}{k-1}} \leq 1. \tag{1}$$

Theorem 2.2 *Let $\mathcal{A} \neq \emptyset$ denote a set of partitions of an $n \geq 2$ element set. Assume that $\mathcal{A}$ does not contain the coarsest partition consisting of a single*