Star chromatic numbers of graphs

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Abstract

We investigate the relation between the star-chromatic number $\chi^*(G)$ and the chromatic number $\chi(G)$ of a graph G. First we give a sufficient condition for graphs under which their star-chromatic numbers are equal to their ordinary chromatic numbers. As a corollary we show that for any two integers k, g, there exists a k-chromatic graph of girth at least g whose star-chromatic number is also k. The special case of this corollary with $g = 4$ answers a question of H. L. Abbott and B. Zhou. We also present an infinite family of triangle free planar graphs with star-chromatic number equal their chromatic number. We then study the star-chromatic number of critical colorable graphs. We prove that if an $(m+1)$-critical graph has large girth, then its star-chromatic number is close to m. We also consider $(m+1)$-critical graphs with high connectivity. An infinite family of graphs is constructed to show that for each $\epsilon > 0$ and each $m \geq 2$ there is an m-connected $(m+1)$-critical graph with star chromatic number at most $m + \epsilon$. This answers another question asked by H. L. Abbott and B. Zhou.

1 Introduction

Let k and d be natural numbers such that $k \geq 2d$. A (k, d)-*coloring* of a graph $G = (V, E)$ is a mapping $\gamma : V \mapsto \mathbb{Z}_k$ such that, for each edge

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$(u,v) \in E$, $|\gamma(u) - \gamma(v)|_k \geq d$, where $|x|_k := min\{|x|, k - |x|\}$. Observe that a $(k,1)$-coloring of a graph G is just an ordinary coloring of G. We say G is (k,d)-*colorable* if there is a (k,d)-coloring of G. The *star chromatic number* $\chi^*(G)$ of G is the minimum $\frac{k}{d}$ for which G is (k,d)-colorable.

The notion of the star chromatic number of a graph was introduced by A. Vince [9]. His work relies on continuous method. A pure combinatorial treatment of star-chromatic number was given by J. A. Bondy and P. Hell [3]. The following alternate definition of the star-chromatic number was given by the second author in [10], and we shall use both definitions in this paper.

Let C be a circle in R^2 of length 1, and let $r \geq 1$ be any real number. Denote by $C^{(r)}$ the set of all open intervals of C of length $1/r$. An r-*circular coloring* of a graph G is a mapping c from $V(G)$ to $C^{(r)}$ such that $c(x) \cap c(y) = \emptyset$ whenever $(x,y) \in E(G)$. If such an r-circular coloring exists, we say that G is r-*circular colorable.*

It was proved in [10] that the star-chromatic number $\chi^*(G)$ of G is equal to the minimum r for which G is r-circular colorable. (To be precise, the "minimum" in the definition of the star-chromatic number should be "infimum". However, it was proved in [9] that the infimum is attained.) Thus we obtain an alternate definition of the star-chromatic number of a graph. This alternate definition is extended by W. Deuber and X. Zhu [5] to weighted graphs and some interesting problems are discussed there.

It follows from the definition that $\chi^*(G) \leq \chi(G)$ for any graph G. There is also a lower bound for $\chi^*(G)$ in terms of $\chi(G)$, namely $\chi(G) - 1 < \chi^*(G)$ (cf. [9]). Thus the parameter $\chi^*(G)$ can be regarded as a refinement of the parameter $\chi(G)$, and it contains more information about the graph. There are graphs G for which $\chi(G) = \chi^*(G)$, and there are also graphs G for which $\chi^*(G)$ are arbitrarily close to $\chi(G) - 1$. A. Vince [9] first asked the question that what determines whether $\chi^*(G) = \chi(G)$. This has been studied by many authors, and some sufficient conditions can be found in [1, 6, 8, 9, 10] under which a graph G has its star-chromatic number equals its ordinary chromatic number. In general, it is intractable to decide whether a given graph satisfies $\chi^*(G) = \chi(G)$, (cf. [8]). In this paper we give another sufficient condition for a graph G to satisfy $\chi^*(G) = \chi(G)$. This condition is weaker (hence the result is stronger) than the previous known sufficient conditions, [1, 6, 8, 9, 10]. As a corollary, we show that uniquely colorable graphs G satisfies $\chi^*(G) = \chi(G)$.