Relaxed coloring of a graph

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Abstract

Let $\mathcal{P}$ be a hereditary family of graphs. A relaxed coloring of a graph $G = (V, E)$ with respect to $\mathcal{P}$ is an assignment of colors to vertices of G so that each color class induces a graph which is the disjoint union of members of $\mathcal{P}$. The $\mathcal{P}$-chromatic number $\chi_{\mathcal{P}}(G)$ of G is the minimum number of colors in a relaxed coloring of G with respect to $\mathcal{P}$. We study the relation between the girth and the $\mathcal{P}$-chromatic number of a graph, and the $\mathcal{P}$-chromatic number of product graphs. Our results generalize some results of M. L. Weaver and D. B. West in [10], and answer some questions in that paper.

1 Introduction

In the last decade, various variations of graph coloring have been studied, [5, 9, 10]. Some of these generalized colorings have requirements stronger than that of the ordinary coloring. In such colorings, the pair of colors assigned to adjacent vertices are not only required to be different, they must also forbidden some patterns. Such colorings can be defined as a homomorphism of a graph G to another graph H, and are usually referred to as H-coloring problems, [1, 2, 5, 6]. Other generalizations have requirements weaker than that of the ordinary coloring. In other words, adjacent vertices are allowed to be colored with the same color. However there are some restrictions on the subgraphs induced by the color classes. Such coloring are called *relaxed colorings* [10] or *improper colorings* [9]. Since this is the context of the present paper, we give a definition.

Let $\mathcal{P}$ be a so called hereditary family of graphs, which is closed under taking subgraphs. A $\mathcal{P}$-k-coloring of a graph $G = (V, E)$ partitions V into k blocks such that the subgraphs induced by the blocks are disjoint unions of members of $\mathcal{P}$. The $\mathcal{P}$-chromatic number $\chi_{\mathcal{P}}(G)$ of G is the minimum number k such that G is $\mathcal{P}$-k-colorable. If $\mathcal{P}$ is the family of empty graphs, then $\chi_{\mathcal{P}}(G) = \chi(G)$. When $\mathcal{P}$ is the family of forests, then $\chi_{\mathcal{P}}(G)$ is also called the *vertex arboricity* of G. For a survey of results concerning $\mathcal{P}$-k-coloring of graphs, readers are referred to [9] and [10].

Following notions in [10], for a given family $\mathcal{H}$ of graphs we denote by $\mathcal{H}^*$ the family consisting of disjoint unions of subgraphs of members of $\mathcal{H}$, and denote by $\mathcal{H}'$ the family of graphs with no subgraph belonging to $\mathcal{H}$. We call the parameters $\chi_{\mathcal{H}^*}(G)$ and $\chi_{\mathcal{H}'}(G)$ the $\mathcal{H}$*-restricted* and the $\mathcal{H}$*-forbidden* chromatic numbers of G, respectively. When $\mathcal{H} = \{H\}$ consists of a single graph, then we use H in place of $\{H\}$. We shall study only forbidden and restricted chromatic numbers. First we discuss the existence of graphs with large girths and large restricted and forbidden chromatic numbers. The corresponding results generalize some results of M. L. Weaver and D. B. West [10]. We then consider the restricted and forbidden chromatic numbers of the Cartesian product $G \square G'$ and the lexicographic product $G[G']$ of graphs. For the Cartesian product, we prove in particular that the product of more than three odd cycles has *path restricted* chromatic number 3. In other words, if