Hammocks and the algorithm of Nazarova-Roiter

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1. *Introduction*

Hammocks have been considered by Brenner [**1**] in order to give a numerical criterion for a finite translation quiver to be the Auslander-Reiten quiver of some representation-finite algebra. Ringel and Vossieck [**11**] gave a combinatorial definition of left hammocks, which generalizes the concept of hammocks in the sense of Brenner, as a translation quiver H and an additive function h on H (called the hammock function) satisfying some conditions, and they showed that a thin left hammock with finitely many projective vertices is just the preprojective component of the Auslander-Reiten quiver of the category of $\mathcal{S}$-spaces, where $\mathcal{S}$ is a finite partially ordered sets (abbreviated: poset). An impotant role in representation theory of poset is played by two differentiation algorithms. One of the algorithms is due to Nazarova and Roiter [**8**] which reduce a poset $\mathcal{S}$ with a maximal element a to a new poset $\mathcal{S}' = {}_a\partial\mathcal{S}$. The second is due to Zavadskij [**13**] which reduce a poset $\mathcal{S}$ with a suitable pair (a, b) of elements a, b to a new poset $\mathcal{S}' = \partial_{(a,b)}\mathcal{S}$. The main purpose of the present paper is to construct new left hammocks from a given one, and to show the relationship between these new left hammocks and the algorithm of Nazarova-Roiter. Our operation on hammocks is a generalization of their algorithm. In a forthcoming paper ([**6**]), We will discuss the relationshipe between hammocks and the algorithm of Zavadskij.

In §2, we recall some basic definitions and facts. Let H be a left hammock with hammock function h_H, $p(a)$ a thin projective vertex of H different from the source. In §3, we construct a new left hammock ${}_aH$ and determine its hammock function $h_{({}_aH)}$. We prove that the subquiver, denote by $H/{}_aH$, consisting of all vertices x satisfying $h_H(x) - h_{({}_aH)}(x) \neq 0$ is also a left hammock. We show that if H is infinite, then ${}_aH$ is also infinite. In case H is a hammock, the relation between the hammock induced by a projective vertex of H and the algorithm of Nazarova-Roiter is stated in §6. The case when $p(a)$ is a non-thin projective vertex of H is discussed in §4. In §5, we consider the hammock induced by some injective vertex of H. The corresponding results concerning $\ell(\mathcal{S})$, the category of $\mathcal{S}$-spaces, are also described.

Throughout this paper, all algebras are assumed to be finite-dimensional (associative) basic algebras with unit over an algebraically closed field k, all modules are finitely generated right modules. We denote by A-mod the category of A-modules. The composition of two morphisms $f : M_1 \to M_2$ and $g : M_2 \to M_3$ is denoted by fg. All posets are assumed to be finite. We denote by $\mathbb{N}, \mathbb{N}_1, \mathbb{Z}$ the set of natural numbers, positive integers and integers, respectively. For all unexplained notation we refer to [**11**] and [**12**].

2. *Preliminaries*

2.1 *Left hammocks and hammocks.* Let $H = (H_0, H_1, \tau)$ be a proper translation quiver. We define inductively the full subquivers ${}^dH$ of H. First of all, ${}^{-1}H$ is the empty quiver, and z belongs to ${}^dH$ if and only if $z^- \subseteq {}^{d-1}H$. Also, ${}^\infty H = \bigcup_{d\in\mathbb{N}} {}^dH$. Thus, for all $d \in \mathbb{N} \cup \{\infty\}$, we see that ${}^dH$ is a predeccessor closed subquiver, and we may consider it as a translation quiver, using the restriction of τ. Suppose H has a unique source ω, and $H = {}^\infty H$, then we define $h_H : H_0 \to \mathbb{Z}$ inductively as follows. By abuse of notation, let $h_H(\tau x) = 0$ for x projective (note that in this case, τx is not defined). Now, let $h_H(\omega) = 1$, and, for $x \neq \omega$, with h_H already defined on all proper predecessors of x, let $h_H(x) = \sum_{y\to x} h_H(y) - h_H(\tau x)$, where y runs over all starting vertices of all arrows ending at x. (note: the multiplicity of arrows $y \to x$ should be calculated in.) With these preparations, we are able to recall the main definition: the translation quiver H is said to be a *left hammock* provided

(1) $H = {}^\infty H$;

(2) H has a unique source ω and $h_H(\omega) = 1$;

(3) h_H takes values in $\mathbb{N}_1$;

(4) if q is an injective vertex then $h_H(q) \geq \sum_{q\to y} h_H(y)$.

When H is a left hammock, the function h_H is said to be its *hammock function.* In fact, $h_H(x)$ is just $\dim_k \mathrm{Hom}_{k(H)}(\omega, x)$ for each $x \in H$.

A vertex x of H is called *thin* if $h_H(x) = 1$. A left hammock H is said to be *thin* provided $h_H(p) = 1$ for any projective vertex p of H. A left hammock H is called a *hammock* if $|H_0| < \infty$. A hammock is always thin and has a unique sink, say ω'.

2.2 *S-spaces.* Fix some field k. Given a poset $\mathcal{S}$, an *$\mathcal{S}$-space* $V = (V_\omega; V_s)_{s\in\mathcal{S}}$ is given by a vector space V_ω over k, and subspaces V_s of V_ω, for $s \in \mathcal{S}$, such that $V_s \subseteq V_t$ for $s \leq t$. We call V_ω the *total space* of V, and define its k-dimemsion by $\underline{\dim}_\omega V = \dim_k V_\omega$. Given two $\mathcal{S}$-spaces V, W, a map $\varphi : V \to W$ is given by a k-linear map $\varphi_\omega : V_\omega \to W_\omega$ satisfying $\varphi_\omega(V_\mathcal{S}) \subseteq W_s$ for all $s \in \mathcal{S}$; the induced map $V_s \to W_s$ will be denoted by φ_s. The posets we will consider are always assumed to be finite. We denote the category of