ON THE LATTICE POINT PROBLEM FOR ELLIPSOIDS

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Abstract. For k-dimensional ellipsoids E with $k \geq 9$ we show that the number of lattice points in rE is approximated by the volume of rE up to an error of order $\mathcal{O}(r^{k-2})$, as r tends to infinity. The estimate is best possible and uniform in the class of ellipsoids with bounded ratios of lengths of axes. It improves the classical estimate $\mathcal{O}(r^{k-2k/(k+1)})$ due to Landau (1915).

1. Introduction and results

Let $\mathbf{R}^k$, $1 \leq k < \infty$ denote a real k-dimensional Euclidean space with scalar product $(\cdot, \cdot)$ and norm $|x|^2 = x_1^2 + \cdots + x_k^2$, for $x = (x_1, \dots, x_k) \in \mathbf{R}^k$. Denote by $|x|_\infty = \max\{|x_j| : \ 1 \leq j \leq k\}$ the maximum-norm. Let $\mathbb{Z}^k$ be the standard lattice of points with integer coordinates in $\mathbf{R}^k$.

For a (measurable) set $B \subset \mathbf{R}^k$ let vol$B$ denote the Lebesgue measure of B, and let $\mathrm{vol}_{\mathbb{Z}}B$ denote the lattice volume of B, that is the number of points in B with integer coordinates.

Let us consider a quadratic form,

$$Q(x) \overset{\text{def}}{=} (Qx, x), \qquad \text{for} \quad x \in \mathbf{R}^k,$$

where $Q : \mathbf{R}^k \to \mathbf{R}^k$ denotes a symmetric positive linear bounded operator, that is $Q(x) > 0$, for $x \neq 0$. Define the ellipsoid

$$E_s \overset{\text{def}}{=} \{x \in \mathbf{R}^k : \ Q(x) \leq s\}, \qquad \text{for} \quad s \geq 0.$$

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We shall prove the following

Theorem 1.1. *For $k \ge 9$,*

$$\Delta(s,Q) \overset{\text{def}}{=} \sup_{a\in\mathbf{R}^k} \left| \frac{\mathrm{vol}_{\mathbb{Z}}(E_s+a)-\mathrm{vol}E_s}{\mathrm{vol}E_s} \right| = \mathcal{O}\left(\frac{1}{s}\right), \qquad \textit{for} \quad s\ge 1.$$

The constant in $\mathcal{O}(s^{-1})$ may depend on k and the lengths of axes of the ellipsoid E_1 only.

An inspection of the proof of this result yields an explicit bound for the remainder term. Let

$$0<q_1^2\le q_2^2\le\cdots\le q_k^2$$

denote the eigenvalues of the operator Q.

Theorem 1.2. *For $k\ge 9$,*

$$\Delta(s,Q)\le c\Gamma\Lambda s^{-1}, \qquad \textit{for} \quad s\ge q_1^2,$$

where $\Lambda = q_1^2(q_k/q_1)^k$ and $\Gamma=(q_k/q_1)^4$, and where the constant c may depend on the dimension k only.

Remark. For the dimension $k=8$ we can prove that

$$\Delta(s,Q)\le cq_8^{12}q_1^{-10}s^{-1}\ln^2(s+1), \qquad \textit{for} \quad s\ge q_1^2$$

with an absolute constant c.

For the class of *general* ellipsoids described above Landau (1915) obtained the estimate $\mathcal{O}(s^{-1+1/(1+k)})$. This result has been extended by Hlawka (1950) to convex bodies with smooth boundary and strictly positive Gaussian curvature. Hlawka's estimate has been recently improved by Krätzel and Nowak (1991,1992) to $\mathcal{O}(s^{-1+\lambda})$, where $\lambda = 5/6k+2$, for $k\ge 8$, and $\lambda = 12/14k+8$, for $3\le k\le 7$.

Since the lower bound $\Delta(s,Q)=\Omega(s^{-1})$ holds for spheres (Q =Identity) (see, e.g. Fricker (1982)), Theorem 1.2 solves the problem of *uniform* error bounds for ellipsoids with arbitrary center provided that the dimension k is sufficiently large, i.e. $k\ge 9$.

The bound of Theorem 1.2 shows that the number of lattice points in an ellipsoid depends asymptotically only on the size of ellipsoid, i.e. only on radii of the largest inscribed and the smallest circumscribed spheres, and does not depend on assumptions like "rationality" or "orientation" of the ellipsoid that is on the conjugation class of Q under the action of $SL(k,\mathbb{Z})$.

For special ellipsoids a number of particular results is available. For example, the error bound $\mathcal{O}(s^{-1})$ holds for $k\ge 5$ for a fixed rational form Q (see Walfisz (1924), $k\ge 9$, and Landau (1924), $k\ge 5$). Jarnik (1928) proved the same bound