Floer Homology and Homotopy Planes

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Abstract

We compute the Floer homology of a certain family of homology 3-spheres occuring as the boundary of a compactification of complex homotopy planes. The Floer complex turns out to be concentrated in even degrees and 2-periodic. This is proved by using suitable bordisms to Seifert fibred spaces.

1 Introduction

A homology plane is a smooth 2-dimensional variety over $\mathbb{C}$, such that its associated analytic space is acyclic. A homotopy plane is a homology plane whose associated analytic space is contractible.

Recently homology planes played some role in the computation of Donaldson polynomials by Fintushel and Stern, using a topological cut and paste technique called rational blow-down ([St]).

The classification problem for homology planes is solved in [tD-Pe] for homotopy planes of logarithmic Kodaira dimension 1. These homotopy planes are distinguished by their minimal compactification divisor. The set of algebraic isomorphism classes of such homotopy planes is parametrised by integers (a, b, m) with a, b relatively prime, $1 < a < b$ and $0 < m$. Let $P(a, b, m)$ be the corresponding homotopy plane. By [Ra] a closed regular neighborhood of a compactification divisor in a suitable algebraic compactification of $P(a, b, m)$ is bounded by a closed smooth oriented 3-manifold $S(a, b, m)$, whose diffeomorphism type is uniquely determined by $P(a, b, m)$. Since $P(a, b, m)$ is contractible, the 3-manifold $S(a, b, m)$ is a homology 3-sphere. Our main theorem is:

Theorem 1.1 *The Floer group* $\mathrm{HF}\,(S(a, b, m))$ *of the homology 3-sphere* $S(a, b, m)$ *is a free abelian group of rank* $\frac{1}{3}m(a^2 - 1)(b^2 - 1)$*, lying in the even part of the grading. It is 4-periodic. If* a, b *are odd it is 2-periodic.*

Our proof is inspired by [St-Sz], where similiar calculations have been made in the special case $a = 2, m = 1$.

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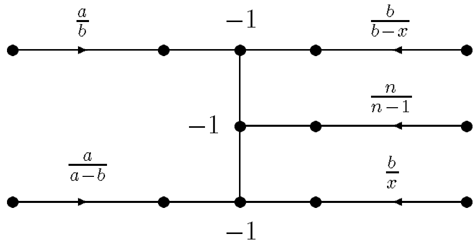
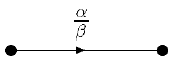
2 Descriptions of the homology 3-spheres

In [tD-Pe] it is shown that the homotopy plane $P(a,b,m)$ allows a smooth algebraic compactification such that the compactification divisor $D(a,b,m)$ has smooth rational irreducible components and normal crossings with intersection graph



. (2.1)

Here x is defined by $-x \equiv a \bmod b$, $0 < x < b$. The symbol



means

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where the $s_i > 1$ are uniquely defined by the continued fraction development

$$\frac{\alpha}{\beta} = [s_n, \ldots, s_1] := s_n - [s_{n-1}, \ldots, s_1]^{-1},\ [\,] := \infty.$$

A tubular neighborhood of the divisor $D(a,b,m)$ is a plumbing of D^2-bundles over 2-spheres according to the $\mathbb{Z}$-weighted graph (2.1), whose boundary $S(a,b,m)$ is diffeomorphic to a Dehn surgery on a framed link in S^3 associated to the $\mathbb{Z}$-weighted graph (2.1).

Let $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\}$. To each $\overline{\mathbb{Q}}$-weighted tree (Γ, w) we associate a $\overline{\mathbb{Q}}$-weighted unoriented link $L = L(\Gamma, w)$ in S^3: Each vertex v of Γ contributes an unknot L_v with weight $w(v)$, such that for each edge $\{v, v'\}$ of Γ the knots L_v and $L_{v'}$ link each other geometrically once.

We collect some necessary remarks on Dehn surgery. For us the term "diffeomorphism" between oriented smooth manifolds always means orientation preserving diffeomorphism unless otherwise stated. Let M be an oriented smooth 3-manifold. Let K be an unoriented knot in M. We also write K for the image of K in M. Let νK be a tubular neighborhood of K. Orient $\partial\nu K$ as the boundary of νK ("outward normal vector first"). A framing of K is an isotopy class of diffeomorphisms $f : S^1 \times S^1 \to \partial\nu K$. A framed knot (M, K, f) yields a diffeomorphism class of an oriented 3-manifold $M(K, f)$, the surgery on (M, K, f),