A Characterization of M–matrices

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Abstract

Since Ostrowski introduced the class of M–matrices more than fifty equivalent definitions of these matrices were established. Here we give another characterization using Schur complements.

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One of the most beautiful classes of structured matrices is the class of M–matrices introduced by Ostrowski in 1937 [O]:

Definition 1.1 *A matrix $A \in \mathbb{R}^{n,n}$ is called a Z–matrix if A can be expressed in the form*

$$A = sI - B,$$

where B is a nonnegative matrix, i.e. $B \geq 0$, $s \in \mathbb{R}$ (here I denotes the identity matrix). If in addition, s is greater than the spectral radius of B ($s > \rho(B)$), A is called M–matrix.

M–matrices arise in many problems in the mathematical, physical, chemical, biological and social sciences. Since 1937 many properties and equivalent definitions of M–matrices were established. The first systematic effort was made by Fiedler

and Pták in 1962 [FP] followed by the work of Varga [V]. The book of Berman and Plemmons gives 50 equivalent definitions for a Z–matrix to be an M–matrix. Here we just mention the following ones

Theorem 1.2 *Let A be a Z–matrix. Then the following properties are equivalent:*

i) A is an M–matrix;

ii) all principal minors of A are positive;

iii) all the leading principal minors of A are positive;

iv) A^{-1} is a nonnegative matrix.

Before we state our new characterization of M–matrices we need well–known definitions and notations.

Notation 1.3 *For $n \in \mathbb{N}$ we write $N := \{1, \ldots, n\}$. If $\alpha, \beta \subseteq N$ we denote by $A_{\alpha\beta}$ the submatrix of $A \in \mathbb{R}^{n,n}$ whose rows are indexed by the elements of α and whose columns are indexed by the elements of β. Moreover, $\bar{\alpha} := N \backslash \alpha$.*
If $\alpha = \{1, \ldots, r\}$ for a $r \in N$, then α and $\bar{\alpha}$ constitute a partition of A. Then we write

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.1}$$

where $A_{11} = A_{\alpha\alpha}$, $A_{12} = A_{\alpha\bar{\alpha}}$ $A_{21} = A_{\bar{\alpha}\alpha}$, $A_{22} = A_{\bar{\alpha}\bar{\alpha}}$. If A_{11} is nonsingular then

$$(A/A_{11}) := A_{22} - A_{21}A_{11}^{-1}A_{12} \tag{1.2}$$

denotes the Schur complement of A with respect to A_{11}.

Theorem 1.4 *Let A be a Z–matrix. Then the following conditions are equivalent:*

1) A is an M–matrix;

2) there exists a partition of A, i.e.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

such that all leading principal minors of A_{11} and (A/A_{11}) are positive;